Notes on the Analytic Hierarchy Process*

by

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Abstract. This progress report outlines the main reasons why Saaty's Analytic Hierarchy Process (AHP, see [17] and [18]) is not a valid methodology.

1. Introduction. We assume that the reader is familiar with the AHP and do not reproduce the methodology here. We also assume that the reader is familiar with the earlier comments on the AHP by Belton and Gear [9] and Dyer [11] which we refer to in sections 11 and 12. A new decision methodology, Preference Function Modeling (PFM), which takes into account the points raised here is described in Barzilai [3, 1].

Section 2 lists the main elements of Miller's Hierarchy Process of which the AHP is a variant. The remaining sections deal with these elements in more detail.

2. The MHP. The concept of decomposition of criteria into a sub-criteria tree (i.e., the generation of operational sub-criteria) was first proposed by Miller in his 1966 doctoral dissertation [13] (see also [14 and 15]) as part of what may properly be named Miller's Hierarchy Process (MHP). Multi-attribute utility theory, proposed by Raiffa in 1969 (see [16, p. 17]), as well as other practical procedures (as opposed to earlier abstract *n*-dimensional mathematical frameworks) for multi-attribute/multi-criteria decision analysis have adopted this very valuable concept. Unfortunately, as we will see below, other aspects of the MHP are the basis of common decision methodologies errors.

The following MHP elements are of interest:

- The construction of a linear *n*-dimensional preference function from one-dimensional components;
- The hierarchical procedure for the numerical computation of criteria weights;
- The interpretation of the weights as relative importance;
- The normalization of weights at each node of the criteria tree; and
- The verbal scales used to elicit weight-ratio estimates.

As is clear from this list, all the major elements of the AHP are present in the MHP although there is no reference to Miller's work in the AHP literature. Since the MHP [13] was published more than ten years before the AHP [17], it is reasonable to conclude that Miller developed his ideas independently of Saaty and to consider the AHP a variant of the MHP. The AHP differs from the MHP in the following ways:

- The AHP has a procedure for reconciling inconsistent user input. This is an important contribution to decision analysis, though the procedure requires a larger than necessary number of estimates. In addition, the use of the eigenvector method to implement the procedure for reconciling inconsistent input is a mathematical error. As a result of this error, the output of this procedure depends on the description of the problem – an unacceptable property of any algorithm.

- Miller's multiple verbal scales have been replaced with a single one. This ostensible improvement limits the flexibility of input elicitation.

3. The MHP/AHP is a Utility Theory. The proponents of the AHP have created the impression that standard frameworks and tools for analyzing decision methodologies do not apply to it. For example, the AHP claims to be different from utility (cf. the title of [22]); to measure priorities as opposed to utilities; and to have its own logic (see the title of [19]). In fact, since the MHP/AHP attaches numbers to alternatives in accordance with their "priorities," the basic principles of measurement theory do apply to it.

The outcomes of measurement procedures depend only on the property being measured and the appropriate degree of uniqueness of the scales involved, i.e., scale type (see Barzilai [5]). Therefore, priorities, utilities and preferences are different labels for the same property and the MHP/AHP is a utility theory – not a valid one, but still, a utility theory.

4. A Linear Preference Function. As early as 1982, Kamenetzky [12] recognized that there is "a relationship" between the AHP and additive utility (or value) functions. In Barzilai [4], we formally proved that the MHP/AHP constructs an n-dimensional (multi-criteria) preference function which is a linear combination of one-dimensional (single-criterion) preference functions.

In the framework of Barzilai [4], each alternative $\mathbf{x}$ is represented by a point $(x_1, \ldots, x_n)$ where x_j is the preference attached to $\mathbf{x}$ on the jth single-criterion and the overall (multi-criteria) preference for $\mathbf{x}$ is given by

$$y = f(\mathbf{x}) = f(x_1, \ldots, x_n) = \sum_{j=1}^{n} w_j x_j . \quad (1)$$

As we will see below, the procedures provided by the MHP/AHP for computing the numbers x_j and the "criteria weights" w_j are flawed.

5. Measurement Units. Denoting overall preference by y and keeping equation (1) in mind, we note (see also Watson and Freeling [23]) that:

- The y and x variables may be measured in arbitrary units. For example, y may be measured in dollars, x_1 in square feet and x_2 in thousands of dollars.
- The coefficients are conversion factors: one unit of the variable x_j is converted into w_j units of the variable y. The dimension of w_j is the dimension of y divided by the dimension of x_j. In the example above, the dimension of w_1 is dollar/square foot.

Therefore:

- The magnitude of the coefficients w_j cannot be determined independently of the x variables (or before the units of the x variables are determined). While the units of the y and x variables may be fixed arbitrarily, the units and magnitudes of the w_j's depend on the units of the x_j's and y.

- If the unit of the variable x_j is changed (multiplied by k) while the unit of y is unchanged, then w_j must be adjusted (divided by k).
- If the unit of the variable y is changed (multiplied by k) while the units of the x variables are all unchanged, then *each* of the conversion factors w_j must be adjusted (divided by k).
- The procedure for determining "criteria weights" in the MHP/AHP *independently of the units* of the single-criterion variables is fundamentally flawed (i.e., by itself, it is sufficient to invalidate the AHP).

6. The "Relative Importance" Interpretation. Churchman and Ackoff [10] discuss a linear model in which the coefficients correspond to the "importance" of the criteria. Accordingly, the MHP, and later the AHP, interpret the coefficient ratio $w_i \div w_j$ as "relative importance." The concept of relative importance is then used to elicit user estimates of coefficient ratios. Churchman and Ackoff, Miller, as well as Saaty do not define what they mean by the terms "importance of criteria" or "relative importance of criteria."

The interpretation of the coefficient ratio $w_i \div w_j$ as relative importance is a fundamental error. Since the magnitude of the coefficient ratio $w_i \div w_j$ depends on the units in which x_i and x_j are measured, it cannot correspond to any reasonable notion of relative importance: if the ith criterion – say the price of a house – is an "important" criterion and x_i is measured in dollars, it cannot become 1000 times "less important" (or simply "unimportant") by changing the x_i unit to thousands of dollars.

7. Normalization. By normalization we mean the multiplication of a set of variables or constants by a single number as is done in various stages of MHP/AHP computations. Normalization is a mathematical operation. As any other mathematical procedure, its use in any methodology must be justified, but this operation has not been justified in the MHP/AHP literature. We note the following:

- The only justification for normalization is a change of unit of the underlying variable.
- Since units of measurement may be assigned arbitrarily, all normalizations are equivalent and there are no "special normalizations." Equivalently, the properties of any valid procedure for preference measurement must be invariant under changes of units or various normalizations.
- Once units of the x variables are fixed, only one normalization, corresponding to a y-variable change of unit, may be performed. This means that the number of degrees of freedom for normalizations on the criteria tree is **one** and that multiplying *all* the w coefficients by a single number is the only admissible normalization.
- The MHP/AHP procedure of normalizing the weights at each node of the criteria tree is a fundamental error.
- In the MHP/AHP, the normalizations of the w coefficients provide constraints which are used to compute these coefficients. When the number of normalizations of coefficients is reduced to one, the remaining information collected by the MHP/AHP is not sufficient to compute these coefficients (see Barzilai [4] for examples and more details).

8. The Eigenvector Procedure. The eigenvector method is not the correct solution to the problem of reconciling inconsistent ratio measurements. This is not a cosmetic or a minor issue.

- When the eigenvector method is applied to this problem, the solution depends on the description of the problem (see Barzilai *et al.* [8] and Barzilai [2]).
- The isomorphism between the ratio and difference estimation problems is a fundamental property of these problems (see Barzilai and Golany [7] and Barzilai [1]). The eigenvector method does not preserve this isomorphism.
- There are procedures (most of them with no foundations) for applying the eigenvector method to a set of incomplete estimates by completing the matrix of ratio estimates. However, it is also desirable in certain situations (for example in the context of the tradeoff problem or for group decision making) to reconcile inconsistent ratio or difference estimates where some of the estimates appear multiple times in the input. The eigenvector method cannot deal with this case.

For a complete review of this issue see Barzilai [2] where we have demonstrated multiple flaws of the use of the eigenvector method in the context of the AHP. We also demonstrated in that paper that the claims in the AHP literature concerning the eigenvector method are incorrect, circular or meaningless.

9. Ratio Scales and Pairwise Comparisons. The AHP claims to measure preference on ratio scales. Since scale transformations of the type $y = qx$ (similarity transformations) are a subset of the affine transformations $y = p + qx$, sets of assumptions which justify measurement of preferences on ratio scales equivalently justify preference measurement on interval scales with the additional constraint $p = 0$. More formally, a necessary (but not sufficient) condition to restrict the affine scale transformations $y = p + qx$ to the subset $y = qx$ is the existence of an absolute zero.

Even in the context of decision/utility theory, this has been understood at least since 1944 – see von Neumann and Morgenstern [24, p. 23]. More explicitly, Stevens [20, p. 679] states: "An absolute zero is always implied, even though the zero value on some scales (e.g. absolute temperature) may never be produced" and "If, in addition, a constant can be added (or a new zero point chosen), it is proof positive that we are not concerned with a ratio scale" [21, p. 29]. Since an absolute zero has not been established (and, in all likelihood, does not exist) for preference measurement, preference cannot be measured on ratio scales.

Note that the AHP claims to measure "intangibles" for which a scale does not pre-exist. It is not reasonable to assume existence of an absolute zero for a scale which does not exist. One has only to consider such familiar variables as *temperature* and *time* to conclude that establishing the existence of an absolute zero for a given property is not a matter of an arbitrary declaration nor is it an easy task.

Furthermore, affine measurement of preference cannot be performed through pairwise comparisons. The smallest number of objects for which measurement on interval scales is not an arbitrary assignment of numbers, is three. In the absence of an absolute zero, the constraint $p = 0$ cannot be imposed and ratio scales are not applicable. For a formal treatment of these and related issues see Barzilai [1].

10. AHP Axioms. Without commenting on the merits of the axiomatic development and foundations of decision methodologies, we can state that the "axioms" underlying the

AHP are meaningless. If they do not properly characterize the AHP, they are of no interest; if they do, they cannot be meaningful either, since they characterize a methodology which suffers from multiple fatal flaws.

11. Various Symptoms. The best known criticism of the AHP is Belton and Gear's [9] "rank reversal" which they proposed to remedy through a change of normalization. This is akin to saying that rank reversal can be avoided by measuring in kilometers instead of miles. It follows from our discussion in §7 that rank reversal is a symptom of deeper problems.

Similarly, other symptoms of deeper problems are the dependence of AHP output on order of operations (cf. Barzilai and Golany [6]) and the generation of non-equivalent preference functions and rankings from equivalent decompositions (e.g. Barzilai [4]) and from equivalent descriptions of subproblems (e.g. Barzilai [2]).

12. Summary. Dyer [11, p. 249] states that the AHP's flaw can be corrected and that "the actual solution to this problem is relatively simple" [11, p. 257]. This erroneous conclusion is based on his claim that the flaw is that rank reversal is a symptom of arbitrary rankings. Unfortunately, these arbitrary rankings are demonstrated through the phenomenon of rank reversal – a circular argument. In fact, the AHP is plagued by *many* flaws and these flaws are fundamental.

Decision methodologies should not be proposed on the basis of identifying and correcting a single error in existing methodologies without a complete understanding of the structure of these methodologies. Of course, no methodology should be considered valid unless it is fully understood, and it is easier to understand the structure of a methodology when all the procedures involved are properly justified and the terminology in use does not obscure its structure.

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