Game Theory Foundational Errors – Part I

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1 The Value of Players

According to Hart [4, p. 210], *value* is the subject of utility theory and is a measure of an outcome to a player in his utility scale, and, "[i]n a similar way, one is interested in evaluating a game; that is, measuring the value of each player in the game."

In current terminology, the relevant issues of measuring value on a utility scale are as follows (see von Neumann and Morgenstern [5, §3] and Barzilai [1, 2]). By an empirical system E we mean a set of empirical objects together with operations and possibly the relation of order which characterize the property under measurement. A mathematical model M of the empirical system E is a set with operations that reflect the operations in E as well as the order in E when E is ordered. A scale s is a homomorphism from E into M, i.e. a mapping of the objects in E into the objects in M that reflects the structure of E into M. The purpose of modelling E by M is to enable the application of mathematical operations on the elements of the mathematical system M. In order for the operations of addition and multiplication to be applicable, the system M must be (i) a field if the empirical system has an absolute *zero* and *one*, (ii) a one-dimensional vector space when the empirical system has an absolute zero but not an absolute one, or (iii) a one-dimensional affine space which is the case for all subjective (i.e. personal or psychological) properties with neither an absolute zero nor absolute one. Affine scales are not unique and the scale $t = p + q \times s$ is equivalent to s. The sum of two points in an affine space is undefined and, therefore, for an affine scale s the expression $s(a) + s(b)$ is undefined. For example, the sum of two times – rather than time differences – is undefined. Since *value* is not a physical property, value scales that enable the operations of addition and multiplication are one-dimensional affine scales. For further details see Barzilai [1].

2 A Sample of Errors

2.1 Undefined Sums

The expression $V(S) + V(T)$, which von Neumann and Morgenstern use in the definition of the characteristic function of a game, has no basis – this sum is undefined for utility scales (see The Principle of Reflection in Barzilai [1]) nor is it defined for affine preference scales. This is the case even for a single player and this is also the case for imputations which are supposed to be utilities.

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2.2 Reduction to a Two-Person Game

2.2.1 "The Value" of a Two-Person Game Is Not Properly Defined

The definition of the characteristic function of a game depends on a reduction to "the value" of a two-person (a coalition vs. its complement) game. This value is not well-defined because utility scales are not unique (see Barzilai [3]).

2.2.2 The Utility of a Coalition

The construction of a two-person-game value depends on the concept of expected utility of a player. The reduction treats a coalition, i.e. a group of players, as a single player but there is no foundation in the theory for *the utility of a group of players.*

2.2.3 The Definition of Value Is Incomplete

Since, by definition, a subjective property is associated with a person, the values of coalitions must be associated with a person whose preference for these coalitions is being evaluated. Without specifying this person the definition of coalition values is incomplete. The literature does not appear to offer an answer to the obvious question – whose value scale is being constructed?

2.2.4 Utility Theory's Shortcomings

Utility theory, which underpins von Neumann and Morgenstern's book [5], is not the correct framework for measuring preference (see Barzilai [2 and 1, §6.4]). *Inter alia,* utility theory does not impose constraints on the values of preference scales for prizes, but the interpretation of the utility operation in terms of lotteries is required in the construction of these scales and this interpretation constrains the values of utility scales for lotteries. The theory permits lotteries that are prizes and this leads to a contradiction since an object may be both a prize, which is not constrained, and a lottery which is constrained.

References

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