On the Mathematical Foundations of Economic Theory

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Abstract

We establish the conditions that must be satisfied for the mathematical operations of linear algebra and calculus to be applicable. The mathematical foundations of the social sciences including economic theory, utility theory, and game theory, depend on these conditions which have not been correctly identified in the classical literature.

Keywords

Foundations of science, economic theory, game theory errors, utility theory, measurement theory, decision theory.

1 Introduction

The construction of the mathematical foundations of any scientific discipline requires the identification of the conditions that must be satisfied in order to enable the application of the mathematical operations of linear algebra and calculus. In addition, the mathematical foundations of social science disciplines, including economic theory, require the application of mathematical operations to *non-physical variables,* i.e, to variables that describe psychological or subjective properties such as *preference*.

Whether psychological properties can be measured (and hence whether mathematical operations can be applied to psychological variables) was debated by a Committee that was appointed in 1932 by the British Association for the Advancement of Science but the opposing views in this debate were not reconciled in the Committee's 1940 Final Report.

In 1944, game theory was proposed as the proper instrument with which to develop a theory of economic behavior where utility theory was to be the means for measuring preference. We show that the interpretation of utility theory's lottery operation which is used to construct utility scales leads to an intrinsic contradiction and that the operations of addition and multiplication are not applicable on utility scale values.

As for game theory, a fundamental concept of this theory is the assignment of values to objects such as outcomes and coalitions, i.e. the construction of value functions. *Value* (or utility, or preference) is not a physical property of the objects being valued, that is, *value* is a subjective (or psychological, or personal) property. Therefore, the definition of *value* requires specifying both *what* is being valued and *whose* values are being measured. Yet *whose* values are being measured in the construction of game theory concepts such as the characteristic function of a game is not specified in the literature. We present additional shortcomings of game theory which render it and its utility underpinning unsuitable to serve as mathematical foundations for economics and we reconstruct these foundations.

2 The Role of Game Theory in Economic Theory

Under the heading "The Mathematical Method in Economics" von Neumann and Morgenstern state in *Theory of Games and Economic Behavior* [11, §1.1.1] that the purpose of the book was "to present a discussion of some fundamental questions of economic theory" and, more specifically, that "... this theory of games of strategy is the proper instrument with which to develop a theory of economic behavior" [11, §1.1.2].

The application of mathematical methods to economic theory requires the applicability of the mathematical tools which these methods employ. Therefore, establishing the applicability of mathematical operations is a prerequisite for a discussion of the mathematical foundations of economic theory. In particular, without the operations of addition and multiplication, which are elementary mathematical tools, very limited results can be derived.

2.1 Measurement of Preference – Empirical Addition

The applicability of mathematical operations is among the issues implicitly addressed by von Neumann and Morgenstern in [11, §§3.2–3.6] in the context of measurement of individual preferences. Preference, or value, or utility, is not a physical property of the objects being valued, that is, preference is a subjective, psychological, property. Whether psychological properties can be measured was an open question in 1940 when a Committee appointed by the British Association for the Advancement of Science in 1932 "to consider and report upon the possibility of Quantitative Estimates of Sensory Events" published its Final Report (Ferguson *et al.* [9]). An Interim Report, published in 1938, included "a statement arguing that sensation intensities are not measurable" as well as a statement arguing that sensation intensities are measurable. These opposing views were not reconciled in the Final Report.

The position that psychological variables cannot be measured was supported by Campbell's view on the role of measurement in physics [7, Part II] which elaborated upon Helmholtz's earlier work on the mathematical modelling of physical measurement [8]. The main elements of this view are summarized by J. Guild in Ferguson *et al.* [9, p. 345] in the context of measurement of *sensation* as follows:

> I submit that any law purporting to express a quantitative relation between sensation intensity and stimulus intensity is not merely false but is in fact meaning-

> less unless and until a meaning can be given to the concept of addition as applied to sensation. No such meaning has ever been defined. When we say that one length is twice another or one mass is twice another we know what is meant: we know that certain practical operations have been defined for the addition of lengths or masses, and it is in terms of these operations, and in no other terms whatever, that we are able to interpret a numerical relation between lengths and masses. But if we say that one sensation intensity is twice another nobody knows what the statement, if true, would imply.

To re-state Guild's position in current terminology the following is needed. By an empirical system E we mean a set of empirical *objects* together with *operations* (i.e. functions) and possibly the relation of *order* which characterize the property under measurement. A mathematical model M of the empirical system E is a set with operations that reflect the empirical operations in E as well as the order in E when E is ordered. A scale s is a mapping of the objects in E into the objects in M that reflects the structure of E into M (in technical terms, a scale is a homomorphism from E into M).

The purpose of modelling E by M is to enable the application of mathematical operations on the elements of the mathematical system M: As Campbell eloquently states [7, pp. 267–268], "the object of measurement is to enable the powerful weapon of mathematical analysis to be applied to the subject matter of science."

In terms of these concepts, Guild states that for psychological variables it is not possible to construct a scale that reflects the empirical operation of addition because such an empirical (or "practical") addition operation has not been defined; if the empirical operation does not exist, its mathematical reflection does not exist as well.

It appears that von Neumann and Morgenstern were aware of Helmholtz, Campbell, and Guild's arguments. As for the *framework of mathematical modelling*, in [11, §3.4.3] they refer to

> ... a mathematical model for the physical domain in question, within which those quantities are defined by numbers, so that in the model the mathematical operation describes the synonymous "natural" operation.

The framework of mathematical modelling is essential. To enable the application of mathematical operations in a given empirical system, the empirical objects are mapped to mathematical objects on which these operations are performed. In mathematical terms, these mappings are functions from the set of empirical objects to the set of mathematical objects (which typically are the real numbers for the reasons given in §4.3). Given two sets, a large number of mappings from one to the other can be constructed, most of which are not related to the characterization of the property under measurement: A given property must be characterized by empirical operations which are specific to this property and these property-specific empirical operations are then reflected to corresponding operations in the mathematical model. Measurement scales are those mappings that reflect the specific empirical operations which characterize the given property to corresponding operations in the mathematical model. Therefore, the construction of measurement scales requires that the property-specific empirical operations be identified and reflected in the mathematical model. Moreover, the operations should be chosen so as to achieve the goal of this construction which is the application of mathematical operations in the mathematical model.

2.2 Empirical Addition – Circumventing the Issue

Accordingly, von Neumann and Morgenstern had to identify the empirical operations that characterize the property of *preference* and construct a corresponding mathematical model. As we shall see in §3, their empirical operation requires an interpretation that leads to an intrinsic contradiction while the operations of addition and multiplication are not enabled in their mathematical model.

The construction of a model for *preference* measurement is addressed by von Neumann and Morgenstern in [11, §3.4] indirectly in the context of measurement of *individual* preference. While the operation of addition as applies to *length* and *mass* results in scales that are unique up to a positive multiplicative constant, physical variables such as *time* and *potential energy* to which standard mathematical operations do apply are unique up to an additive constant and a positive multiplicative constant. (If s and t are two scales then for *time* or *potential energy* $t = p + q \times s$ for some real numbers p and $q > 0$ while for *length* or *mass* $t = q \times s$ for some $q > 0$.) This observation implies that Guild's argument against the possibility of measurement of psychological variables is not entirely correct. It also seems to indicate the need to identify an empirical – "natural" in von Neumann and Morgenstern's terminology – operation for *preference* measurement for which the resulting scales are unique up to an additive constant and a positive multiplicative constant. Seeking an empirical operation that mimics the "center of gravity" operation, they identified the now-familiar utility theory's operation of constructing lotteries on "prizes" to serve this purpose.

Von Neumann and Morgenstern's *uniqueness* argument and *center of gravity* operation are the central elements of their utility theory which is formalized in the axioms of [11, §3.6]. This theory is the basis of game theory which, in turn, was to serve as the mathematical foundation of economic theory. Elaborating upon von Neumann and Morgenstern's concepts, Stevens [13] proposed a uniqueness-based classification of "scale type" and research interest turned from the issues of the possibility of measurement of psychological variables and the applicability of mathematical operations on scale values to the construction of "interval" scales, i.e. scales that are unique up to an additive constant and a positive multiplicative constant.

3 Shortcomings of Utility and Game Theory

Guild's argument against the possibility of measurement of psychological variables can be rejected on the basis of the uniqueness argument but constructing utility scales that are immune from this argument is not equivalent to establishing that psychological variables can be measured. As we now show, the operations of addition and multiplication do not apply to utility scale values.

This and additional shortcomings of utility theory render it unsuitable to serve as the foundation for applying mathematical methods in economic theory (see also Barzilai [5 and 6]). As for game theory, we will replace its utility foundations with proper ones but additional corrections are required if it is to be the proper instrument with which to develop a theory of economic behavior (see also Barzilai [2–4]).

3.1 Applicability of Operations on Scale Values vs. Scale Operations

Consider the applicability of the operations of addition and multiplication to scale values for a fixed scale, that is, operations that express facts such as "the weight of an object equals the sum of the weights of two other ones" (which corresponds to addition: $s(a) = s(b) + s(c)$) and "the weight of a given object is two and a half times the weight of another" (which corresponds to multiplication: $s(a) = 2.5 \times s(b)$).

It is important to emphasize the distinction between the application of the operations of addition and multiplication to scale values for a fixed scale (for example $s(a) = s(b) + s(c)$) as opposed to what appear to be the same operations when they are applied to an entire scale whereby an equivalent scale is produced (for example $t = p + q \times s$ where s and t are two scales and p, q are numbers). In the case of scale values for a fixed scale, the operations of addition and multiplication are applied to elements of the mathematical system M and the result is another element of M. In the case of operations on entire scales, addition or multiplication by a number are applied to an element of the set $S = \{s, t, \ldots\}$ of all possible scales and the result is another element of S rather than M. These are different operations because operations are functions and functions with different domains or ranges are different.

In the case of "interval" scales where the uniqueness of the set of all possible scales is characterized by scale transformations of the form $t = p + q \times s$, it cannot be concluded that the operations of addition and multiplication are applicable to scale values for a fixed scale such as $s(a) = s(b) + s(c)$. It might be claimed that the characterization of scale uniqueness by $t = p + q \times s$ implies the applicability of addition and multiplication to scale values for fixed scales, but this claim requires proof. (There is no such proof, nor such claim, in the literature because a simple argument[1] shows that this claim is false.)

3.2 The Principle of Reflection and the Utility Operation

3.2.1 The Principle of Reflection

Consider the measurement of *length* and suppose that we can only carry out ordinal measurement on a set of objects, that is, for any pair of objects we can determine which one is longer or whether they are equal in length (in which case we can order the objects by their length). This may be due to a deficiency with the state of technology (appropriate tools are not available) or with the state of science (the state of knowledge and understanding of the empirical or mathematical system is insufficient). We can still construct scales (functions) that map the empirical objects into the real numbers but

1 Consider the automorphisms of the group of integers under addition. The group is a model of itself ($E = M$), and scale transformations are multiplicative: $t = (\pm 1) \times s$. However, by definition, the operation of multiplication which is defined on the set of scales is not defined on the group M.

although the real numbers admit many operations and relations, the only relation on ordinal scale values that is relevant to the property under measurement is the relation of order. Specifically, the operations of addition and multiplication can be carried out on the range of such scales since the range is a subset of the real numbers, but such operations are extraneous because they do not reflect corresponding empirical operations. Extraneous operations may not be carried out on scale values – they are irrelevant and inapplicable; their application to scale values is a modelling error.

The Principle of Reflection is an essential element of modelling that states that operations within the mathematical system are applicable *if and only if* they reflect corresponding operations within the empirical system. In technical terms, in order for the mathematical system to be a valid model of the empirical one, the mathematical system must be homomorphic to the empirical system (a homomorphism is a structure-preserving mapping). A mathematical operation is a valid element of the model only if it is the homomorphic image of an empirical operation. Other operations are not applicable on scale values.

By the Principle of Reflection, a necessary condition for the applicability of an operation on scale values is the existence of a corresponding empirical operation (the homomorphic pre-image of the mathematical operation). That is, the Principle of Reflection applies in both directions and a given operation is applicable in the mathematical image only if the empirical system is equipped with a corresponding operation.

3.2.2 Addition and Multiplication Are Not Applicable to Utility Scales

The Principle of Reflection implies that the operations of addition and multiplication are not enabled on utility scales despite their "interval" type. These operations are not applicable to von Neumann and Morgenstern's utility model because their axioms include *one* compound empirical *ternary* operation (i.e. the "center of gravity" operation which is a function of *three* variables) rather than the *two binary* operations of addition and multiplication (each of which is a function of *two* variables). Addition and multiplication are not enabled on utility scale values in later formulations as well because none of these formulations is based on two empirical operations that correspond to addition and multiplication. It should be noted that the goal of constructing the utility framework was to enable the application of mathematical operations rather than to build a system with a certain type of uniqueness.

3.3 Utility Theory's Intrinsic Inconsistency

As an abstract mathematical system, von Neumann and Morgenstern's utility axioms are consistent. However, while von Neumann and Morgenstern establish the *existence* and *uniqueness* of scales that satisfy these axioms, they do not address utility scale *construction.* This construction requires an interpretation of the empirical operation in the context of preference measurement and although the axioms are consistent in the abstract, the *interpretation* of the empirical utility operation creates an intrinsic inconsistency: The interpretation of the utility operation in terms of lotteries constrains the values of utility scales for lotteries while the values of utility scales for prizes are unconstrained; the theory permits lotteries that are prizes and this leads to a contradiction since an object may be both a prize, which is not constrained, and a lottery which is constrained.

3.4 Game Theory Values

The assignment of values to objects such as outcomes and coalitions, i.e. the construction of value functions, is a fundamental concept of game theory. *Value* (or utility, or preference) is not a physical property of the objects being valued, that is, *value* is a subjective (or psychological, or personal) property. Therefore, the definition of *value* requires specifying both *what* is being valued and *whose* values are being measured.

Game theory's characteristic function assigns values to coalitions but von Neumann and Morgenstern do not specify *whose* values are being measured in the construction of this function. Since it is not possible to construct a value (or utility) scale of an unspecified person or a group of persons, game theory's characteristic function is not well-defined. Likewise, all game theory concepts that do not specify *whose* values are being measured are ill-defined.

3.5 Undefined Sums

The expression $v(S)+v(T)$ which represents the sum of coalition values in von Neumann and Morgenstern's definition of the characteristic function of a game has no basis since, by *The Principle of Reflection,* addition is undefined for utility or value scales. The sum of points on a straight line in an affine geometry, which is the correct model for preference measurement (see §4.1), is undefined as well. For the same reasons, the sum of imputations, which are utilities, is undefined.

3.6 The Utility of a Coalition

The definition of the characteristic function of a game depends on a reduction to "the value" of a two-person (a coalition vs. its complement) game. In turn, the construction of a two-person-game value depends on the concept of expected utility of a player. The reduction treats a coalition, i.e. a group of players, as a single player but there is no foundation in the theory for *the utility of a group of players.*

3.7 "The Value" of a Two-Person Zero-Sum Game Is Ill-Defined

To construct von Neumann and Morgenstern's characteristic function, a coalition and its complement are treated as players in a two-person zero-sum game, and the coalition is assigned its "single player" value in this reduced game. However, the concept of "the value" of two-person zero-sum game theory is not unique and consequently is ill-defined.

The minimax theorem which states that every two-person zero-sum game with finitely many pure strategies has optimal mixed strategies is a cornerstone of game theory. Given a two-person zero-sum game, denote by x^* and y^* the minimax optimal strategies and by u the utility function of player 1. Utility functions are not unique and for any p and positive q, u is equivalent to $p+q\times u$ but since the minimax optimal strategies do not depend on the choice of p and q, x^* and y^* are well-defined. However, the value of the game varies when p and q vary so that it depends on the choice of the utility function u and given an arbitrary real number v, the numbers p and q can be chosen so that the value of the game equals v. As a result, the concept of "the value" of a game is ill-defined and any game theoretic concept that depends on "the" value of a game is ill-defined as well.

4 The Reconstruction of the Foundations

4.1 Proper Scales – Straight Lines

In order to enable the "powerful weapon of mathematical analysis" to be applied to economic theory or any scientific discipline it is necessary, at a minimum, to construct models that enable the operations of addition and multiplication for without these operations the tools of linear algebra and elementary statistics cannot be applied. This construction, which leads to the well-known geometry of points on a straight line, is based on two observations:

1. If the operations of addition and multiplication are to be enabled in the mathematical system M, these operations must be defined in M. The empirical system E must then be equipped with corresponding operations in order for M to be a model of E.
2. Mathematical systems with an absolute *zero* or *one* are not homogeneous: these special, distinguishable, elements are unlike others. On the other hand, since the existence of an absolute *zero* or *one* for empirical systems that characterize subjective properties has not been established, they must be modelled by homogeneous mathematical systems.

Sets that are equipped with the operations of addition and multiplication, including the inverse operations of subtraction and division, are studied in abstract algebra and are called *fields*. The axioms that define fields are listed in *Appendix A*. A field is a not a homogeneous system since it contains two special elements, namely an absolute *zero* and an absolute *one* which are the additive and multiplicative identities of the field (in technical terms, they are invariant under field automorphisms). It follows that to model a subjective property by a mathematical system where the operations of addition and multiplication are defined we need to modify a field in order to homogenize its special elements, i.e., we need to construct a *homogeneous field*. To homogenize the multiplicative identity, we construct a one-dimensional vector space which is a *partially homogeneous field* (it is homogeneous with respect to the multiplicative identity but not with respect to the additive identity) where the elements of the field serve as the set of scalars in the vector space. To homogenize the additive identity as well, we combine points with the vectors and scalars and construct a one-dimensional affine space, which is a homogeneous field, over the previously constructed vector space. The axioms characterizing vector and affine spaces are listed in *Appendix A*. The end result of this construction, the one-dimensional affine space, is the algebraic formulation of the familiar straight line of elementary (affine) geometry so that for the operations of addition and multiplication to be enabled on models that characterize subjective properties, the empirical objects must correspond to points on a straight line of an affine geometry. For details see *Appendix A*, or the equivalent formulations in Artzy [1, p. 79], and Postnikov [12, pp. 46–47].

In an affine space, the difference of two points is a vector and no other operations are defined on points. In particular, it is important to note that the ratio of two points as well as the sum of two points are undefined. The operation of addition is defined on *point differences,* which are vectors, and this operation satisfies the *group* axioms listed in *Appendix A*. Multiplication of a vector by a scalar is defined and the result is a vector. In the one-dimensional case, and only in this case, the ratio of a vector divided by another non-zero vector is a scalar.

Returning to Guild's argument as quoted in §2.1, we note that the operations of addition and multiplication have different forms in fields, one-dimensional vector spaces, and one-dimensional affine spaces. Although his argument is correct with respect to the application of *The Principle of Reflection* and the identification of addition as a fundamental operation, that argument does not take into account the role of the multiplication operation and the modified forms of addition and multiplication when the models correctly account for the degree of homogeneity of the relevant systems. Note also that it is not sufficient to model the operation of addition since, except for the natural numbers, multiplication is not repeated addition: In general, and in particular for the real numbers, multiplication is not defined as repeated addition but through field axioms.

Since the purpose of modelling is to enable the application of mathematical operations, we classify scales by the type of mathematical operations that are enabled on them. We use the terms *proper scales* to denote scales where the operations of addition and multiplication are enabled on scale values, and *weak scales* to denote scales where these operations are not enabled. This partition is of fundamental importance and we note that it follows from *The Principle of Reflection* that all the models in the literature of economic theory and classical measurement theory are weak because they are based on operations that do not correspond to addition and multiplication.

4.2 Implications: Undefined Ratios and Pairwise Comparisons

In order for the operations of addition and multiplication to be applicable, the mathematical system M must be (i) a field if it is a model of a system with an absolute *zero* and *one*, (ii) a one-dimensional vector space when the empirical system has an absolute *zero* but not an absolute *one*, or (iii) a one-dimensional affine space which is the case for all non-physical properties with neither an absolute *zero* nor absolute *one*. This implies that for proper scales, scale ratios are undefined for subjective variables including *preference*. In particular, this invalidates all decision methodologies that apply the operations of addition and multiplication to scale values and are based on preference ratios. For example, in the absence of an absolute zero for *time,* it must be modelled as a homogeneous variable and the ratio of two times (the expression t_1/t_2), is undefined. For the same reason, the ratio of two potential energies e_1/e_2 is undefined while *the ratios of the differences* $\Delta t_1/\Delta t_2$ and $\Delta e_1/\Delta e_2$ are properly defined. In Section §3.5 we saw that the sum of von Neumann and Morgenstern's utility scale values is undefined. Since the sum of two points in an affine space is undefined, the sum of proper preference scale values is undefined as well.

The expression $(a-b)/(c-d) = k$ where *a, b, c, d* are points on an affine straight line and k is a scalar is used in the construction of proper scales. The number of points in the left hand side of this expression can be reduced from four to three (e.g. if $b = d$) but it cannot be reduced to two and this implies that pairwise comparisons cannot be used to construct scales where the operations of addition and multiplication are enabled.

4.3 Strong Scales – the Real Numbers

Proper scales enable the application of the operations of linear algebra but are not necessarily equipped with the relation of order which is needed to indicate a direction on the straight line (for example, to indicate that an object is more preferable, or heavier, or more beautiful than another). To construct proper ordered scales the underlying field must be ordered (for example, the field of complex numbers is unordered while the field of the rational numbers is ordered). For a formal definition of an ordered field see McShane and Botts [10, Ch. 1, §3].

Physics, as well as other sciences, cannot be developed without the mathematical "weapons" of calculus. For example, the basic concept of acceleration in Newton's Second Law is defined as a (second) derivative; in statistics, the standard deviation requires the use of the square root function whose definition requires the limit operation; and marginal rates of change, defined by partial derivatives, are used in economics. If calculus is to be enabled on ordered proper scales, the underlying field must be an ordered field where any limit of elements of the field is itself an element of the field. In technical terms, the underlying field must be *complete* (see McShane and Botts [10, Ch. 1, §5] for a formal definition). Since the only ordered complete field is the field of real numbers, in order to enable the operations of addition and multiplication, the relation of order, and the application of calculus on subjective scales, the objects must be mapped into the real, ordered, homogeneous field, i.e. a one-dimensional, real, ordered, affine space, and the set of objects must be a subset of points on an empirical ordered real straight line. We use the term *strong models* to denote such models and *strong scales* to denote scales produced by strong models.

The application of the powerful weapon of mathematical analysis requires a system in which addition and multiplication, order, and limits are enabled. The reason for the central role played by the real numbers in science is that the field of real numbers is the only ordered complete field.

5 Summary

We identified the conditions that must be satisfied in order to enable the application of linear algebra and calculus, and established that there is only one model for strong measurement of subjective variables. When these conditions, which have not been correctly identified in the literature, are satisfied, the operations of addition and multiplication are applicable to scale values representing non-physical variables. The mathematical foundations of the social sciences need to be corrected to account for these conditions.

Appendix A: The Axioms of an Affine Straight Line

Groups and Fields

A *group* is a set G with an operation that satisfies the following requirements (i.e. axioms or assumptions):

- The operation is *closed:* the result of applying the operation to any two elements a and b in G is another element c in G. We use the notation $c = a \circ b$ and since the operation is applicable to pairs of elements of G, it is said to be a binary operation.
- The operation is *associative:* $(a \circ b) \circ c = a \circ (b \circ c)$ for any elements in G.
- The group has an *identity:* there is an element e in G such that $a \circ e = a$ for any element a in G.
- *Inverse elements:* for any element a in G, the equation $a \circ x = e$ has a unique solution x in G. If $a \circ x = e$, x is called the inverse of a.

If $a \circ b = b \circ a$ for all elements of a group, the group is called *commutative*. We reemphasize that a group is an algebraic structure with *one* operation and we also note that a group is not a homogeneous structure because it contains an element, namely its identity, which is unlike any other element of the group since the identity of a group G is the only element of the group that satisfies $a \circ e = a$ for all a in G. (The identity is the only fixed point of all the group's automorphisms.)

A *field* is a set F with two operations that satisfy the following assumptions:

- The set F with one of the operations is a commutative group. This operation is called *addition* and the identity of the additive group is called zero (denoted '0').
- The set of all non-zero elements of F is a commutative group under the other operation on F. That operation is called *multiplication* and the multiplicative identity is called one (denoted '1').
- For any element a of the field, $a \times 0 = 0$.
- For any elements of the field the *distributive* law $a \times (b + c) = (a \times b) + (a \times c)$ holds.

Two operations are called addition and multiplication only if they are related to one another by satisfying the requirements of a field; a single operation on a set is not termed addition nor multiplication. The additive inverse of the element a is denoted $-a$, and the multiplicative inverse of a non-zero element is denoted a^{-1} or $1/a$. Subtraction and division are defined by $a - b = a + (-b)$ and $a/b = a \times b^{-1}$.

Vector and Affine Spaces

A vector space is a pair of sets (V, F) together with associated operations as follows. The elements of F are termed *scalars* and F is a field. The elements of V are termed *vectors* and V is a commutative group under an operation termed vector addition. These sets and operations are connected by the additional requirement that for any scalars

$j, k \in F$ and vectors $u, v \in V$ the scalar product $k \cdot v \in V$ is defined and satisfies, in the usual notation, $(j+k)v = jv + kv$, $k(u+v) = ku + kv$, $(jk)v = j(kv)$ and $1 \cdot v = v$.

An *affine space* is a triplet of sets (P, V, F) together with associated operations as follows. The pair (V, F) is a vector space. The elements of P are termed *points* and two functions are defined on points: a one-to-one and onto function $h : P \to V$ and a "difference" function $\Delta : P^2 \to V$ that is defined by $\Delta(a, b) = h(a) - h(b)$. Note that this difference mapping is not a closed operation on P: although points and vectors can be identified through the one-to-one correspondence $h : P \to V$, the sets of points and vectors are equipped with different operations. Specifically, the operations of addition, multiplication, and division are not defined on points. If $\Delta(a, b) = v$, it is convenient to say that the difference between the points a and b is the vector v. Accordingly, we say that an affine space is equipped with the operations of (vector) addition and (scalar) multiplication *on point differences*. Note that in an affine space no point is distinguishable from any other.

The dimension of the affine space (P, V, F) is the dimension of the vector space V. By a homogeneous field we mean a *one-dimensional* affine space. A homogeneous field is therefore an affine space (P, V, F) such that for any pair of vectors $u, v \in V$ where $v \neq 0$ there exists a unique scalar $\alpha \in F$ so that $u = \alpha v$. In a homogeneous field (P, V, F) the set P is termed a *straight line* and the vectors and points are said to be collinear. Division in a homogeneous field is defined as follows. For $u, v \in V$, $u/v = \alpha$ means that $u = \alpha v$ provided that $v \neq 0$. Therefore, in an affine space, the expression $\Delta(a, b)/\Delta(c, d)$ for the points $a, b, c, d \in P$ where $\Delta(c, d) \neq 0$, is defined and is a scalar:

$$\frac{\Delta(a, b)}{\Delta(c, d)} \in F \tag{1}$$

if and only if the space is one-dimensional, i.e. it is a straight line or a homogeneous field. When the space is a straight line, $\Delta(a, b)/\Delta(c, d) = \alpha$ (where $\Delta(c, d) \neq 0$) means by definition that $\Delta(a, b) = \alpha\Delta(c, d)$.

References

[1] R. Artzy, *Linear Geometry,* Addison-Wesley, 1965.

[2] J. Barzilai, "Game Theory Foundational Errors and the Mathematical Foundations of Operations research" *CORS-SCRO Bulletin*, Vol. 41 No. 4, pp. 42–43, November 2007. Posted at www.scientificmetrics.com

[3] J. Barzilai, "Game Theory Foundational Errors – Part I," Technical Report, Dept. of Industrial Engineering, Dalhousie University, pp. 1–2, 2007. Posted at www.scientificmetrics.com

[4] J. Barzilai, "A Game Theory Pitfall – "The Value of a Game"," *CORS-SCRO Bulletin*, Vol. 41 No. 3, p. 44, July 2007. Posted at www.scientificmetrics.com

[5] J. Barzilai, "On the Mathematical Modelling of Measurement," arxiv.org, math.GM/0609555, pp. 1–4, 2006. Posted at www.scientificmetrics.com

[6] J. Barzilai, "Preference Modeling in Engineering Design," in *Decision Making in Engineering Design,* K.E. Lewis, W. Chen and L.C. Schmidt (Eds.), ASME Press, pp. 43–47, 2006.

[7] N.R. Campbell, *Physics the Elements,* Cambridge University Press, 1920.

[8] H. von Helmholtz, "Zählen und Messen erkenntnis-theoretisch betrachet," *Philosophische Aufsätze,* Fues's Verlag, pp. 17–52, 1887. English translation by C.L. Bryan, *Counting and Measuring,* van Nostrand, 1930.

[9] A. Ferguson, C.S. Myers, R.J. Bartlett, H. Banister, F.C. Bartlett, W. Brown, N.R. Campbell, K.J.W. Craik, J. Drever, J. Guild, R.A. Houstoun, J.O. Irwin, G.W.C. Kaye, S.J.F. Philpott, L.F. Richardson, J.H. Shaxby, T. Smith, R.H. Thouless and W.S. Tucker, "Quantitative Estimates of Sensory Events," *Advancement of Science,* Vol. 2, pp. 331–349, 1940.

[10] E.J. McShane and T.A. Botts, *Real Analysis,* van Nostrand, 1959.

[11] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior,* Princeton University Press, 1944.

[12] M. Postnikov, *Analytic Geometry* (Lectures in Geometry, Semester I), Mir Publishers, 1982.

[13] S.S. Stevens, "On the Theory of Scales of Measurement," *Science*, Vol. 103, pp. 677–680, 1946.