Game Theory Foundational Errors – Part II

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Abstract

Preference measurement plays a fundamental role, and is necessary in order to introduce the real numbers and operations on them, in game theory and economics. It is not possible to escape the need to construct preference functions by assuming that payoffs are in money units and that each player has a utility function which is linear in terms of money. The mathematical operations of game theory are performed on preferences for objects rather than on empirical objects, preference scales are not unique, and preference spaces are not vectors spaces. Undefined sums and ill-defined concepts which are based on preference scales where *whose preference is being measured* is not specified are the source of many game theory errors. For example, the characteristic function of a game, the Shapley value, the concept of imputations, and von Neumann and Morgenstern's solution of a game are ill-defined. Von Neumann and Morgenstern's utility theory and its variants cannot serve as the foundations of any scientific discipline because their construction rules are contradictory and for additional reasons.

1 Introduction

In Barzilai [5] we listed the following foundational game theory errors:

1. The use of undefined sums.
2. The concept of *"the" value of a two-person game* which is used in the reduction of a coalitional game to a two-person game is ill-defined.
3. There is no basis for *the utility of a group of players* which is used in the reduction of a coalitional game to a two-person game.
4. For a game in "coalitional form" the concept of *value* (and consequently the very definition of a game) is ill-defined because *whose values are being constructed* is not specified.
5. Utility theory cannot serve as the foundation for game theory or any scientific discipline because it is flawed in multiple ways.

In addition to the discussion in Barzilai [4], the following should be noted. The *construction* of the characteristic function of a game is ignored in the literature where it is assumed that a characteristic function is "given" and conclusions are drawn from its numerical values. This is not surprising since without specifying whose values are being measured the characteristic function of a game cannot be constructed.

All game theory concepts that depend on values where it is not specified whose values are being measured are ill-defined (see e.g. Barzilai [3]). This includes Shapley's value [20, 11, and Chapter 3 in 2] and its variants and generalizations (e.g. McLean [17], Monderer and Samet [18], and Winter [22]). Moreover, since the current definition of an n-person game employs the ill-defined concept of the characteristic function (see e.g. Monderer and Samet [18, p. 2058]), the very definition of a game has no foundation.

2 The Essential Role of Preference

Under the heading "The Mathematical Method in Economics" von Neumann and Morgenstern state in *Theory of Games and Economic Behavior* [19, §1.1.1] that the purpose of the book was "to present a discussion of some fundamental questions of economic theory." *The role of preference measurement in game theory is essential* because the outcomes of economic activity are empirical objects rather than real numbers such as $\sqrt{\pi}$ and the application of mathematical operations such as addition and multiplication requires the mathematical modelling of economic systems by corresponding mathematical systems (see Barzilai [4, p. 3] for a detailed discussion). In other words, *the purpose of preference measurement is to introduce the real numbers and operations on them in order to enable the application of The Mathematical Method.*

Consider Guild's statement in support of the position that mathematical operations are not applicable to non-physical variables[1] as summarized in Ferguson *et al.* [8, p. 345] in the context of measurement of *sensation*:

> I submit that any law purporting to express a quantitative relation between sensation intensity and stimulus intensity is not merely false but is in fact meaningless unless and until a meaning can be given to the concept of addition as applied to sensation. No such meaning has ever been defined. When we say that one length is twice another or one mass is twice another we know what is meant: we know that certain practical operations have been defined for the addition of lengths or masses, and it is in terms of these operations, and in no other terms whatever, that we are able to interpret a numerical relation between lengths and masses. But if we say that one sensation intensity is twice another nobody knows what the statement, if true, would imply.

Note that the *property* (length, mass, etc.) of the objects must be specified in order for the mathematical operations to be applicable and that addition and multiplication are

1 Guild's position as well as the opposing position were based on incorrect arguments concerning the applicability of mathematical operations to non-physical variables – see Barzilai [4, p. 9] for details.

applied on lengths and masses of objects. It is not possible to "add objects" without knowing whether what is being added is their mass, length, temperature, etc. Observing that *preference* is the only property of relevance in the context of the mathematical foundations of game theory, we conclude that preference measurement is not a cosmetic issue but a fundamental one in this context.

2.1 Implications

The fact that *preference modelling is of the essence* in game theory implies that much of the theory is in error. Under the title "What is game theory trying to accomplish?" Aumann [1] says that game theory is not a branch of abstract mathematics but is rather motivated by and related to the world around us. As pointed out above, economic transactions are not performed in order to attain as an outcome the number $\sqrt{\pi}$. Stated differently, the outcome of a real-world economic transaction is seldom a real number. One therefore cannot simply "assume" (see e.g. definition 2.3 in Aumann [2]) that the outcome of an economic transaction which is modelled as a play of a game is a numerical payoff function. The only way to introduce the real numbers, and thereby *The Mathematical Method,* into game theory is through the construction of preference functions which represent preference for empirical objects including outcomes of games. As we shall see in §4, it is not possible to escape the need to construct preference functions by "assuming that payoffs are in money units and that each player has a utility function which is linear in terms of money" (Aumann [2, p. 106]). Note that this statement implies that utility is a property of money so that in the fundamental structure of preference modelling (see Barzilai [4, p. 3]), money, in the form of a $20 bill, or twenty coconuts, cocoa beans, dried fish, salt bars, or a beaver pelt (cf. Shubik [21, p. 361]), is an object rather than a property of empirical objects. In the context of mathematical modelling the distinction between objects and properties of objects is fundamental.[2]

Having concluded that the mathematical operations of game theory are performed on preferences for objects rather than on empirical objects, recall that (i) preference functions are not unique (they are unique up to affine transformations), and (ii) the sum of values of a preference function is undefined (see Barzilai [4, §3.5]).

At this point it is necessary to note that scale construction for physical variables requires the specification of the property under measurement. For example, if the property under measurement is *temperature*, the construction results in a *temperature* scale and, clearly, the measurement of *length* does not produce a *mass* scale. In the case of subjective measurement too, the property under measurement must be explicitly specified. If the property under measurement is *preference*, the resulting scales are *preference* scales. Noting that von Neumann and Morgenstern's measurement of preference [19, §3.1] results in utility scales, we conclude that *preference* and *utility* (and, for the same reason, *value,* and *worth*) are synonyms for the same underlying subjective property. It

2 In addition to these considerations, the mathematical operations of game theory must be performed on the preferences of the players because what matters to them is their preferences for the outcomes rather than the physical outcomes.

follows that the distinction between utility theory and value theory has no foundation in logic and science. For example, Keeney and Raiffa's notion of "the utility of value" ($u[v(x)]$, in [14, p. 221]) is as meaningless as "the temperature of the temperature of water" or "the length of the length of a pencil" are.

Some of the errors listed in §1 are due to the fact that the sum of values of a preference function is undefined (utility or preference spaces are not vector spaces). In particular, it follows that the concept of *imputations* is ill-defined because it depends on an undefined sum (e.g. Aumann [2, Definition 4.5, p. 38]). In consequence, throughout the literature of game theory, the treatment of the topic of the division of the "payoff" among the players in a coalition has no foundation.

3 Utility Theory's Shortcomings

3.1 Von Neumann and Morgenstern's Utility

The fundamental role of preference modelling in game theory was recognized by von Neumann and Morgenstern (see [19, §§3.5–3.6]) but their treatment of this difficult problem which is the basis for later developments in "modern utility theory" (cf. Fishburn [9, §1.3] and Coombs *et al.* [6, p. 122]), suffers from multiple flaws and this theory cannot serve as a foundation for any scientific theory.

In essence, von Neumann and Morgenstern study a set of objects A equipped with an operation (i.e. a function) and order that satisfy certain assumptions. The operation is of the form $f(\alpha, a, b)$, where a and b are objects in A, α is a real number, and $c = f(\alpha, a, b)$ is an object in A. Their main result is an *existence* and *uniqueness* theorem for scales (homomorphisms) that reflect the structure of the set A into a set B equipped with order and a corresponding operation $g(\alpha, s(a), s(b))$ where $a \to s(a)$, $b \to s(b)$, and $f(\alpha, a, b) \to g(\alpha, s(a), s(b)$.

3.2 Barzilai's Paradox: Utility's Intrinsic Self-Contradiction

As an abstract mathematical system, von Neumann and Morgenstern's utility axioms are consistent. However, while von Neumann and Morgenstern establish the *existence* and *uniqueness* of scales that satisfy these axioms, they do not address utility scale *construction.* This construction requires a specific interpretation of the empirical operation in the context of preference measurement and although the axioms are consistent in the abstract, *the interpretation of the empirical utility operation*[3] *creates an intrinsic contradiction.* Utility theory constrains the values of utility scales for lotteries while the values of utility scales for prizes are unconstrained. The theory permits lotteries that are prizes (cf. Luce and Raiffa's "neat example" [16, pp. 26–27]) and this leads to a contradiction since an object may be both a prize, which is not constrained, and a lottery which is constrained. In other words, utility theory has one rule for assigning values to prizes

3 In terms of lotteries. An empirical interpretation of the operation is required for this construction.

and a different, conflicting, rule for assigning values to lotteries. For a prize which is a lottery ticket, the conflicting rules are contradictory.

3.2.1 A Numerical Example

Suppose the prizes A and C are assigned by a decision maker the utility values $u(A) = 0$, and $u(C) = 1$ and let D be the lottery $D = \{(0.6, A);(0.4, C)\}$. Utility theory dictates the value $u(D) = 0.6u(A) + 0.4u(C) = 0.4$ (see e.g. Keeney and Raiffa [14]) so that $u(D)$ is determined by the other given parameters and the decision maker has no discretion as to its value.

Now suppose that the decision maker assigns the value $u(B) = 0.5$ to a prize B, and is offered an additional prize E. According to utility theory, there are no constraints on the possible utility values for prizes so that the value of $u(E)$ is at the discretion of the decision maker and is not dictated by the theory. The decision maker then assigns the utility value $u(E) = 0.8$.

Since utility theory allows prizes that are lottery tickets, suppose that the prize E is the lottery ticket $E = \{(0.6, A);(0.4, C)\}$. It follows that $D = E$ yet the utility value of this object is either 0.8 or 0.4 depending on whether we label the object $\{(0.6, A);(0.4, C)\}$ a prize or a lottery. That is, we have $u(D) = 0.4 \neq 0.8 = u(E)$ where D and E are the same object! In other words, the utility value of the object $\{(0.6, A);(0.4, C)\}$ depends on its label. Note that $u(D) < u(B)$ and $u(E) > u(B)$ yet $D = E$ so that the object $\{(0.6, A);(0.4, C)\}$ is rejected in favour of B if it is labelled a lottery ticket and accepted as preferred to B if it is labelled a prize.

3.3 Utility Theory is Neither Prescriptive Nor Normative

Coombs *et al.* [6, p. 123]) state that "utility theory was developed as a prescriptive theory." This claim has no basis since von Neumann and Morgenstern's utility theory as well as its later variants (e.g. Luce and Raiffa [16, §2.5], Fishburn [9, pp. 7–9], Coombs *et al.* [6, pp. 122–129], French [10, Ch. 5], Luce [15, p. 195]) are mathematical theories. These theories are of the form $P \rightarrow Q$, that is, *if* the assumptions P hold *then* the conclusions Q follow. In other words, these theories are not of the form *"Thou Shalt Assume P"* but rather "*if* you assume P." Since mathematical theories do not dictate to decision makers what sets of assumptions they *should* satisfy, the claim that utility theory is prescriptive has no basis in mathematical logic nor in modern utility theory.

Howard says that a normative theory establishes norms for how things should be (*In Praise of the Old Time Religion* [13, p. 29]) and appears to suggest that decision theory says how you should act in compliance with von Neumann and Morgenstern's assumptions [13, p. 31]. His comments on "second-rate thinking" and education [13, p. 30] seem to indicate that he believes that those who do not share his praise for the old time utility religion need to be re-educated. In the context of logic and science this position is untenable – mathematical theories do not dictate assumptions to decision makers. Furthermore, educating decision makers to follow flawed theories is not a remedy for "second-rate thinking." Flawed theories should be corrected rather than be taught as the norm.

Unfortunately, according to Edwards [7, pp. 254–255]), Howard is not alone. Edwards reports as editor of the proceedings of a conference on utility theories that the attendees of the conference unanimously agreed that the experimental and observational evidence has established as a fact the assertion that people do not maximize "subjective expected utility" and the attendees also unanimously stated that they consider "subjective expected utility" to be the appropriate normative rule for decision making under risk or uncertainty. These utility theorists are saying that although decision makers reject the assumptions of the *mathematical theory* of utility, they should accept the conclusions which these assumptions imply. This position is logically untenable.

3.4 Von Neumann and Morgenstern's Structure is Not Operational

The construction of utility functions requires the interpretation of the operation $f(\alpha, a_1, a_0)$ as the construction of a lottery on the prizes a_1, a_0 with probabilities α, $1 - \alpha$ respectively. The utility of a prize a is then assigned the value α where $u(a_1) = 1$, $u(a_0) = 0$ and $a = f(\alpha, a_1, a_0)$.

In order for $f(\alpha, a_1, a_0)$ to be an operation, it must be a single-valued function. Presumably with this in mind, von Neumann and Morgenstern interpret the relation of equality on elements of the set A as *true identity:* in [19, A.1.1-2, p. 617] they remark in the hope of "dispelling possible misunderstanding" that "[w]e do not axiomatize the relation =, but interpret it as *true identity*." If equality is interpreted as true identity, equality of the form $a = f(\alpha, a_1, a_0)$ cannot hold when a is a prize since a lottery and a prize are not identical objects. Consequently, von Neumann and Morgenstern's interpretation of their axioms does not enable the practical construction of utility functions.

Possibly for this reason, later variants of utility theory (e.g. Luce and Raiffa [16]) interpret equality as indifference rather than true identity. This interpretation requires the extension of the set A to contain the lotteries in addition to the prizes. In this model, lotteries are elements of the set A rather than an operation on A so that this extended set is no longer equipped with any operations but rather with the relations of order and indifference (see e.g. Coombs *et al.* [6, p. 122]). This utility structure is not homomorphic (and therefore is not equivalent) to von Neumann and Morgenstern's structure and the utility functions it generates are weak (i.e. do not enable the operations of addition and multiplication) and only enable the relation of order despite their "interval" type of uniqueness.

3.5 All Utility Models are Weak

Although modern utility models (e.g. Luce and Raiffa [16, §2.5], Fishburn [9, pp. 7–9], Coombs *et al.* [6, pp. 122–129], French [10, Ch. 5]) are not equivalent to von Neumann and Morgenstern's model, *The Principle of Reflection* (see Barzilai [4, §3.2.1]) implies that all utility models are weak: despite the fact that they produce "interval" scales, none of these models enables the operations of addition and multiplication (see Barzilai [4, §3.2]).[4]

4 On "Utility Functions that Are Linear in Money"

Consider again the assumption that "payoffs are in money units and that each player has a utility function which is linear in terms of money" (Aumann [2, p. 106]). In addition to the obvious reasons for rejecting this assumption,[5] we re-emphasize that money is not a property of objects and preference functions are unique up to affine rather than linear transformations. This implies that in the case of monetary outcomes it is still necessary to construct the decision maker's preference function for money.

It is correct to say that a given decision maker (who must be identified since preference is a subjective property) is indifferent between the objects A and B where B is a sum of money, which means that $f(A) = f(B)$ where f is the decision maker's preference function. However, the statement that the outcome of a play is the object A and $f(A) = f(B)$, requires the determination of the preference value $f(A)$ and, *in addition,* $f(B)$ as well as the identification of the object B for which $f(A) = f(B)$. It follows that this indirect and more laborious procedure does not eliminate the need to construct the decision maker's preference function and *game theory cannot be divorced from preference modelling*.

5 Errors Not Corrected

It has been suggested that the errors uncovered here have been corrected in recent times, but this is not the case. In the Preface to his 1989 *Lectures on Game Theory* [2], Aumann states that its material has not been superseded. This material includes a discussion of game theory without its preference underpinning, the use of undefined sums, ill-defined concepts, and additional errors.

For example, the "payoff functions" h^i in part (3) of Definition 2.3 in Aumann [2] are not unique and there is no basis for assuming that the outcomes of games are real numbers. Moreover, these functions are unique up to additive and multiplicative constants which are not independent of the index i. As a result, the very definition of a game has no basis even in the simplest two-person case. In the absence of the property of preference, no operations are applicable in game theory but when preference is modelled the sum of values of a preference function is undefined. Such sums appear in

4 Addition and multiplication are not applicable in von Neumann and Morgenstern's utility model because their axioms include *one* compound empirical *ternary* operation (i.e. the "center of gravity" operation which is a function of *three* variables) instead of the *two binary* operations of addition and multiplication (each of which is a function of *two* variables). Addition and multiplication are not enabled on utility scale values in later formulations as well because none of these formulations is based on two empirical operations that correspond to addition and multiplication. It should be noted that the goal of constructing the utility framework was to enable the application of mathematical operations rather than to build a system with a certain type of uniqueness.

5 E.g. the St. Petersburg Paradox which implies that this is an unrealistic assumption. Note that it is necessary to make the even more unrealistic assumption that the additive and multiplicative constants in the players' utility scales are all identical.

Aumann [2] (Definition 3.9 p. 28, Definitions 4.3 and 4.6 p. 38), and throughout game theory's literature.

While Aumann's discussion of Shapley's value ignores utility theory altogether, Hart introduces his 1989 *Shapley Value* [11] as an evaluation, "in the [sic] participant's utility scale," of the prospective outcomes. He then refers explicitly to utility theory and to measuring the value of each player in the game. Note that in addition to the use of undefined sums and ill-defined concepts in the context of Shapley's value, it is not clear whether Shapley's value is intended to represent the evaluation of prospective outcomes of a game by a player or the evaluation of the players themselves (not surprisingly, the question who evaluates the players is not addressed in the literature).

More recently Hart (2004, [12, pp. 36–37]), denoting by x^i the utility of an outcome to player i, refers to the set of utilities as the set of *feasible payoff vectors* and uses the undefined sum of these utilities $\sum_{i \in S} x^i$ in the definition of a "transferable utility" game. As pointed out earlier, utility spaces are not vector spaces.

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