Applicability of Mathematical Operations on Value Scales

Overview

The construction of value scales is of fundamental importance in decision analysis, game theory, and other branches of operations research. The concept of *value* and its measurement also underlies the mathematical foundations of game theory and microeconomics and it plays an essential role in measurement theory and mathematical psychology.

Whether non-physical variables can be measured in order to apply mathematical operations on them was an open question as late as 1940 following a lengthy scientific debate. The controversy appeared to have been settled by von Neumann and Morgenstern in 1944 in their *Theory of Games and Economic Behavior*. However, rather than establish the applicability of mathematical operations on non-physical variables, they have addressed the unrelated problem of scale uniqueness which, following Stevens's extension [1946] of von Neumann and Morgenstern's work, has since been known as the problem of scale classification. As a result, the applicability of mathematical operations has not been further investigated in the literature. The conditions for applicability of mathematical operations have not been identified and they have been applied incorrectly and without foundation throughout the literature of decision theory, the theory of games, microeconomics, measurement theory, and elsewhere.

Addition and multiplication, the operations of fields and vector spaces, are not applicable on ordinal, ratio, or difference scale values. The mathematical space where addition and multiplication are applicable for variables that have no absolute zero has not been identified in the literature. The correct space is a one-dimensional affine space which is the case for many physical variables e.g. time and potential energy and for all non-physical variables including *value*. Geometrically, a one-dimensional affine space is a straight line on which no point, including origin and unit, is marked. The sum and the ratio of two points on such a line are undefined. The formal definition of an affine space is given below.

Scale Definition – the Framework

An empirical system E is a set of empirical objects together with operations, and possibly the relation of order, which characterize a property under measurement. A mathematical model M of the empirical system E is a set with operations that reflect the operations in E as well as the order in E when E is ordered. A scale s is a homomorphism from E into M, i.e. a mapping of the objects in E into the objects in M that reflects the structure of E into M. The purpose of modelling E by M is to enable the application of mathematical operations on the elements of the mathematical system M and mathematical operations in M are applicable if and only if they reflect empirical operations in E.

The framework of mathematical modelling is essential because it is the only way by which mathematical operations can be introduced into decision theory, the theory of games, economic theory, or any other theory. To enable the application of mathematical operations, the empirical objects are mapped to mathematical objects on which these operations are performed. In mathematical terms, these mappings are functions from the set of empirical objects to the set of mathematical objects. Given two sets, a large number of mappings from one to the other can be constructed, most of which are not related to the characterization of the property under measurement: A given property must be characterized by empirical operations which are specific to this property and these property-specific empirical operations are then reflected to corresponding operations in the mathematical model. Measurement scales are those mappings that reflect the specific empirical operations which characterize the given property to corresponding operations in the mathematical model. Therefore, the construction of measurement scales requires that the property-specific empirical operations be identified and reflected in the mathematical model.

Scale construction for physical variables requires the specification of the set of objects and the property under measurement as well as the applicable operations on these objects. Since *value* is not a physical property of the objects being valued, the construction of scales for *value* and other non-physical properties (which are also referred to as personal, psychological, or subjective), requires also the specification of the evaluator. For example, the characteristic function of game theory is ill-defined because in this case the evaluator is not specified [Barzilai 2008].

Applicability of Operations: Mathematical Spaces

Mathematical spaces, e.g. vector or metric spaces, are sets of objects on which specific relations and operations (i.e. functions or mappings) are defined. They are distinguished by these relations and operations while the objects are arbitrary.

Only those relations and operations that are defined in a given mathematical space are relevant and applicable when that space is considered: The application of undefined operations is an error. For example, although the operations of addition and multiplication are defined in the *field* of real numbers, multiplication is undefined in the *group* of real numbers under addition and is not applicable in this group.

Ordinal, Vector, and Affine Spaces

An ordinal space is a set A of objects equipped only with the relations of order and equality. Since order and equality are not operations, i.e. single-valued functions, no operations are defined in ordinal spaces. Specifically, the operations of addition and multiplication (and their inverses – subtraction and division) are not applicable in ordinal spaces.

The formal definitions of vector and affine spaces that follow restrict the operations that are applicable in these spaces. Vectors may be added and subtracted but their multiplication is restricted to the mixed scalar-vector multiplication. The product and ratio of two vectors are undefined except that in the one-dimensional case, and only in this case, the ratio of two vectors is a scalar. The only operation on affine points is the difference operation where the difference of two affine points is a vector.

The Axioms of an Affine Straight Line

Groups and Fields

A *group* is a set G with an operation that satisfies the following requirements (i.e. axioms or assumptions):

- The operation is *closed:* the result of applying the operation to any two elements a and b in G is another element c in G. We use the notation $c = a \circ b$ and since the operation is applicable to pairs of elements of G, it is said to be a binary operation.
- The operation is *associative:* $(a \circ b) \circ c = a \circ (b \circ c)$ for any elements in G.
- The group has an *identity:* there is an element e of G such that $a \circ e = a$ for any element a in G.
- *Inverse elements:* for any element a in G, the equation $a \circ x = e$ has a unique solution x in G. If $a \circ x = e$, x is called the inverse of a.

If $a \circ b = b \circ a$ for all elements of a group, the group is *commutative.* A group is an algebraic structure with *one* operation and is not a homogeneous structure because it contains an element, namely its identity, which is unlike any other element of the group since the identity of a group G is the only element of the group that satisfies $a \circ e = a$ for all a in G.

A *field* is a set F with two operations that satisfy the following assumptions:

- The set F with one of the operations is a commutative group. This operation is called *addition* and the identity of the additive group is called zero (denoted '0').
- The set of all non-zero elements of F is a commutative group under the other operation on F. That operation is called *multiplication* and the multiplicative identity is called one (denoted '1').
- For any element a of the field, $a \times 0 = 0$.
- For any elements of the field the *distributive* law $a \times (b + c) = (a \times b) + (a \times c)$ holds.

Two operations are called addition and multiplication only if they are related to one another by satisfying the requirements of a field; a single operation on a set is not termed addition nor multiplication. The additive inverse of the element a is denoted $-a$, and the multiplicative inverse of a non-zero element is denoted a^{-1} or $1/a$. Subtraction and division are defined by $a - b = a + (-b)$ and $a / b = a \times b^{-1}$.

A field F is ordered if it contains a subset P such that if $a, b \in P$, then $a + b \in P$ and $a \times b \in P$, and for any $a \in F$ exactly one of $a = 0$, or $a \in P$, or $-a \in P$ holds.

Vector and Affine Spaces

A vector space is a pair of sets (V, F) together with associated operations as follows. The elements of F are termed *scalars* and F is a field. The elements of V are termed *vectors* and V is a commutative group under an operation termed vector addition. These sets and operations are connected by the additional requirement that for any scalars $j, k \in F$ and vectors $u, v \in V$ the scalar product $k\,v \in V$ is defined and satisfies, in the usual notation, $(j + k)v = jv + kv$, $k(u + v) = ku + kv$, $(jk)v = j(kv)$ and $1\,v = v$.

An *affine space* is a triplet of sets (P, V, F) together with associated operations as follows. The pair (V, F) is a vector space. The elements of P are termed *points* and two functions are defined on points: a one-to-one and onto function $h : P \to V$ and a function $\Delta : P^2 \to V$ that is defined by $\Delta(a, b) = h(a) - h(b)$. This "difference" mapping is not a closed operation on P: although points and vectors can be identified through the one-to-one correspondence $h : P \to V$, the sets of points and vectors are equipped with different operations and the operations of addition and multiplication are not defined on points. If $\Delta(a, b) = v$, it is convenient to say that the difference between the points a and b is the vector v. Accordingly, we say that an affine space is equipped with the operations of (vector) addition and (scalar) multiplication *on point differences*. In an affine space no point is distinguishable from any other.

The dimension of the affine space (P, V, F) is the dimension of the vector space V. By a homogeneous field we mean a *one-dimensional* affine space. A homogeneous field is therefore an affine space (P, V, F) such that for any pair of vectors $u, v \in V$ where $v \neq 0$ there exists a unique scalar $\alpha \in F$ so that $u = \alpha v$. In a homogeneous field (P, V, F) the set P is termed a *straight line* and the vectors and points are said to be collinear. Division in a homogeneous field is defined as follows. For $u, v \in V$, $u / v = \alpha$ means that $u = \alpha v$ provided that $v \neq 0$. Therefore, in an affine space, the expression $\Delta(a, b) / \Delta(c, d)$ for the points $a, b, c, d \in P$ where $\Delta(c, d) \neq 0$, is defined and is a scalar:

$$\frac{\Delta(a, b)}{\Delta(c, d)} \in F \tag{1}$$

if and only if the space is one-dimensional, i.e. it is a straight line or a homogeneous field. When the space is a straight line, $\Delta(a, b) / \Delta(c, d) = \alpha$ (where $a, b, c, d \in P$) means by definition that $\Delta(a, b) = \alpha\Delta(c, d)$.

Implications – Ratio Scales and the AHP

The Analytic Hierarchy Process (AHP) is a methodology for constructing multi-criteria preference scales. As is the case for other methodologies, the operations of addition and multiplication are not applicable on AHP scale values. The applicability of addition and multiplication must be established *before* these operations are used to compute AHP eigenvectors and the fact that eigenvectors are unique up to a multiplicative constant does not imply the applicability of addition and multiplication.

In order for addition and multiplication to be applicable on preference scale values, the alternatives must correspond to points on a straight line in an affine space. Since the ratio of points on an affine straight line is undefined, preference ratios, which are the building blocks of AHP scales, are undefined. Pairwise comparisons cannot be used to construct affine straight lines.

Implications – Mathematical Economics

Expected Utility Scales. An expected utility space is equipped with two ordered sets: The set of "prizes" and the set of the real numbers in the interval (0, 1). The only operation in this space is the mixed "lottery" operation. This operation is a function of at least three variables (two prizes and a probability). A binary operation is a function of two variables and a ternary operation is a function of three variables. Since expected utility spaces are equipped with *one ternary* operation rather than *two binary* operations that correspond to addition and multiplication, the operations of addition and multiplication are not defined and are not applicable on expected utility scale values. In other words, addition and multiplication are the operations of fields and vector spaces. Since expected utility spaces are neither fields nor vector spaces, addition and multiplication are not applicable in expected utility spaces.

Ordinal Utility Scales. An ordinal space, i.e. an ordered set, is not a Euclidean space. Since it is not a vector space, the elementary operations of addition and multiplication are not applicable in an ordinal space and the operations and concepts of algebra and calculus are undefined in ordinal spaces. In particular, norms, metrics, derivatives, and convexity are undefined and are not applicable in an ordinal space. Therefore, ordinal utility functions are not differentiable and, conversely, differentiable scales cannot be ordinal.

Since the partial derivatives of an ordinal utility function do not exist, the concept of marginal utility is undefined in an ordinal space. Differentiation of ordinal scales in modern demand theory and elsewhere in microeconomics is founded on errors that have no parallel in mathematics and science: Thermodynamics is not, and cannot be, founded on ordinal temperature scales. Clearly, the concept of "slope" is undefined on an *ordinal* topographic map.

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