Slutsky's Mathematical Economics

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Abstract
In a 1915 paper Slutsky addresses the problem of applicability of mathematical methods to economic theory in the context of the modern theory of value. His treatment is founded on fundamental errors.

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Mathematical Economics, Mathematics of the Social Sciences, Utility Functions, Theory of Value

1 Introduction

Whether and which mathematical operations are applicable on scale values for physical and non-physical, i.e. psychological, variables is a foundations-of-science question. The treatment of this problem in Barzilai [2 and 1, §3.2] is summarized in Section 2.

In Section 3 we show that Slutsky's treatment of this problem in his 1915 paper [12] is founded on fundamental errors: The mathematical operations of economic theory, including those he applies in his paper, have no basis; Economic theory cannot avoid the question of applicability of mathematical operations on psychological variables as he suggests; And ordinal data are not sufficient for applying differential calculus operations. These errors are amplified rather than corrected in the current literature of economic theory.

2 Preliminaries

2.1 Terminology

An n-ary relation on a set A is a subset of the Cartesian product A^n. A function from X to Y is a relation on $X \times Y$ that satisfies $[(x, y_1) \in f] \wedge [(x, y_2) \in f] \Rightarrow y_1 = y_2$ (in common notation, $[(f(x) = y_1) \wedge (f(x) = y_2)] \Rightarrow y_1 = y_2)$. An operation is a function, i.e. a single-valued relation. For details and examples see Roberts [10, Ch. 1].

Given the sets L and R and the relations $l_i \in L$ and $r_i \in R$ of the same size k_i for $i = 1, 2, \ldots$, if there exists a function s from L to R such that for all the elements of L

$$(a_1, a_2, \ldots, a_{k_i}) \in l_i \text{ if and only if } (s(a_1), s(a_2), \ldots, s(a_{k_i})) \in r_i \tag{1}$$

then s is termed a homomorphism, the relations $\{r_i\}$ constitute a model of the relations $\{l_i\}$, and s is a structure-preserving mapping that reflects the relations in L into corresponding relations in R. It should be emphasized that "R is a model of L" only in the sense that the relations $\{r_i\}$ are models of the relations $\{l_i\}$.

Clearly, $\{l_i\}$ must be defined in order for a homomrphism to reflect the relation $\{l_i\}$ into $\{r_i\}$, and no conclusions concerning the structure of L can be drawn from relations in R that are not reflections of corresponding relations in L.

2.2 Measurement

Measurement is the process of scale construction for the variables of science and the social disciplines. Measurement scales are those mappings that reflect the specific empirical operations which characterize a given property to corresponding operations in a mathematical model. The construction of measurement scales requires that these property-specific empirical operations be identified and reflected in the mathematical model.

In this context, E is a set of empirical objects together with *operations* and possibly the relation of *order,* which characterize a property (e.g. weight or temperature) of the empirical objects, and a scale s is a homomorphism from E into a mathematical system M. The scale s reflects the structure of E into M and M is then a model of the empirical system E.

This framework, which is due to Helmholtz [4], and is the only basis for the introduction of mathematics into any discipline, has been universally accepted since 1887. See for example Pareto [9, Ch. III, §§55–58, 1906], Campbell [3, Ch. X, 1920], von Neumann and Morgenstern [8, §3.4, 1944], Krantz et al. [6, pp. 8–9 and Chs. 2–4, 1971], Roberts [10, §2.1, 1979], and Barzilai [1, §3.2, 2010].

2.3 The Purpose of Measurement

The purpose of modelling E by M is to enable the application of mathematical operations on the elements of the mathematical system M: As Campbell says, "the object of measurement is to enable the powerful weapon of mathematical analysis to be applied to the subject matter of science" [3, pp. 267–268]. This framework is essential as there is no other basis for applying mathematical operations in science or the social disciplines.

2.4 Applicability of Operations

For $i = 1, 2, \ldots$, given empirical operations $\{e_i\}$ on a set E, if there exists a homomorphism (i.e. a scale) s from E to M that reflects $\{e_i\}$ to $\{m_i\}$, then the operations $\{m_i\}$ are applicable on the scale values $s(a)$ for $a \in E$. Furthermore, these are the only operations that are enabled by this homomorphism. In other words, in a model M of a system E the only operations that are applicable are those which are images of corresponding operations in E. This implies the following principle.

2.5 The Principle of Reflection

Given empirical operations $\{e_i\}$ on a set E of objects, the operations $\{m_i\}$ in M are applicable on the scale values $s(a)$ for $a \in E$ if and only if s reflects $\{e_i\}$ to $\{m_i\}$ (see Barzilai [1, §3.3]).

In particular, if the only empirical relation in E is order and s is a scale that reflects this order into an ordered set M (typically the ordered real numbers), then the operations of addition and multiplication are not applicable on scale values. In this case E is an ordinal system and the only relation on scale values is order.

3 Slutsky's Theory of Value

3.1 The Framework

Slutsky's treatment of the question of applicability of mathematical methods to economic theory – to which he refers in the first sentence of his paper [12, p. 27] – is based on fundamental errors. Economic theory cannot avoid the question of applicability of mathematical operations on psychological variables as he suggests because the basic variable of the *theory of value* – to which he also refers in the first sentence of his paper – is *value*, and *value* is not a physical property of objects.

Furthermore, the mathematical operations of economic theory have no basis when applied in a mathematical system that is disconnected from a corresponding empirical system, yet Slutsky says that these systems "do not seem to [him] to be closely related" [12, p. 28].

In addition, Slutsky's notion that the question of applicability of mathematical operations on psychological variables in economic theory can be avoided is contradicted when he applies the mathematical operations of differential calculus on utility functions (see [12, p. 29]). "Utility" as well as "desire" and "preference" which appear in his definition of a utility function and the sentence following it [12, §2, p. 28] are non-physical variables, i.e. psychological ones. What Slutsky calls "a function index of utility" is, according to his own definition of a utility function, constructed through a homomorphism reflecting the empirical system to a mathematical one in contradiction of his view that the empirical and mathematical systems are not closely related. The mathematical system *must be* a model of the empirical one.

3.2 The Ordinal Claim

Since order is the only relation in Slutsky's definition of utility [12, §2, p. 28], no operations are applicable on his utility functions. His utility space is ordinal, "the second derivatives of the utility function" [12, §2, p. 29–30] are undefined, and the discussion that follows has no foundation.

Hicks amplifies Slutsky's errors in his 1939 *Value and Capital* [5, p. 19]. But while he states that his theory is essentially Slutsky's, Slutsky *ignores the empirical system* of which the mathematical one is a model, whereas Hicks *purges the mathematical system* by undertaking "a purge, rejecting all concepts which are tainted by quantitative utility." This claimed purge of all concepts which are tainted by quantitative utility is contradicted by an (incorrect) analysis of the ratio of marginal utilities, i.e., by applying the mathematical operation of division on quantitative partial derivatives of utility functions (in an ordinal space).

Following Slutsky and Hicks, Samuelson [11, pp. 94–95] adds to their errors. Applying the tools of differential calculus where the assumptions for their applicability are not satisfied, he purportedly proves that marginal utility ratios can be derived from ordinal data.

For a detailed analysis of these and related errors see Barzilai [1 and 2]. More than a hundred years after the publication of Slutsky's paper, these errors are still part of mathematical economics (see e.g. Mas-Colell et al. [7, p. 71]).

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