On Microeconomics Errors and Ordinal Group Decision Making

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1 The Foundational Errors

Microeconomics is founded on mathematical errors that have been committed by Pareto [15] and Hicks [9], and amplified by Samuelson [17], Debreu [7], von Neumann and Morgenstern [14], and those who followed them, including Mas-Colell [11]. These persistent and systematic mathematical errors have been propagated or committed independently in game theory, decision theory and, in fact, throughout the social sciences. The errors are logical and technical such as applying mathematical operations where they are not applicable, proving the wrong theorems, applying conclusions where the assumptions that lead to these conclusions are not satisfied, misidentifying the relevant mathematical spaces, using ill-defined concepts, misinterpreting the meaning of assumptions (axioms) and conclusions, and more – see Barzilai [2–6] for details.

In the classical literature the mathematical models of measurement are incorrect even for the elementary properties of mass and length (see e.g. Krantz et al. [10], Roberts [16], or Narens [13]); the role of the property under measurement is not understood and, in some important cases, what property is being measured is not understood.

2 The Framework: Applicability of Mathematical Operations on Non-Physical Scale Values

The construction of the mathematical foundations of any scientific discipline requires the identification of the conditions that must be satisfied in order to enable the application of the mathematical operations of linear algebra and calculus. In addition, the mathematical foundations of social science disciplines, including economic theory, require the application of mathematical operations to *non-physical variables,* i.e, to variables that describe psychological or subjective properties such as *utility* or *preference.*

Whether psychological properties can be measured, and hence whether mathematical operations can be applied to psychological variables, remained an open question when in 1940 a Committee appointed by the British Association for the Advancement of Science in 1932 "to consider and report upon the possibility of Quantitative Estimates of Sensory Events" published its Final Report. An Interim Report, published in 1938, included "a statement arguing that sensation intensities are not measurable" as well as a statement arguing that sensation intensities are measurable. These opposing views were not reconciled in the 1940 Final Report (see Ferguson et al. [8]).

In *Theory of Games and Economic Behavior* [14, 1944] von Neumann and Morgenstern proposed game theory as "the proper instrument with which to develop a theory of economic behavior." Since the operations of addition and multiplication are applicable on some scales but not on ordinal scales, some mathematical conditions must be satisfied for these operations to be applicable – addition and multiplication cannot be applied without a foundation. These conditions have not been identified and are not satisfied by any scales constructed in the classical literature, including von Neumann and Morgenstern's utility scales. For this and additional reasons, game theory cannot serve as the foundations of economic theory or other disciplines. For a detailed analysis see Barzilai [2–3].

When the conditions for applicability of addition and multiplication on non-physical variables are satisfied, these variables are represented by points in one-dimensional affine spaces. Although vector-space operations are not applicable in affine spaces, they are applied, incorrectly, throughout the literature of economics, theory of games, decision theory, and other disciplines. For example, potential energy, which does not have an absolute zero, is an affine – rather than a vector-space – variable and the sum of two potential energies is undefined. The same holds for "utility" or "value" scales: the sum of "utilities" $u(x) + u(y)$ is undefined not only for different persons but also for a single person using a single fixed scale. The operation of addition is applicable on *differences* of potential energy, or time, or position. Similarly, the ratios of affine variables (again, such as potential energy) are undefined, essentially because there is no absolute zero in an affine space. Since non-physical variables such as preference, utility, or value do not have an absolute zero, preference ratios are undefined. Such undefined ratios are also used in decision analysis.

3 The Ordinal Utility Claim in Economic Theory

The claim that *ordinal temperature scales are sufficient to carry out partial differentiation in thermodynamics* is obviously false. For the same reasons that the mathematical theory of thermodynamics cannot be founded on ordinal temperature scales, modern economic theory cannot be founded on ordinal data.

Since the operations of addition and multiplication are not applicable on ordinal scale values, ordinal utility functions cannot be differentiated – the derivatives (ordinal "marginal utilities") do not exist. Conversely, since ordinal scales cannot be differentiated, differentiable scales are not ordinal. To emphasize, utility scales that satisfy differential conditions are not ordinal: when "marginal utilities" exist, the utility scales in question cannot be ordinal.

The notion that modern economic theory can be founded on ordinal utility theory is an error. For example, the equilibrium conditions of demand theory are stated in terms of "marginal utilities" which are partial derivatives of utility functions. If the utility scales of demand theory are ordinal they are not differentiable; if they are differentiable they cannot be ordinal.

The claim that modern economic theory can be founded on ordinal utility theory appears throughout the literature of modern economic theory (see for example Samuelson [17] and Mas-Colell et al. [12]). The source of this claim is an error in a brief comment by Pareto [15, Appendix, §§5–6] that was amplified by Hicks [9] and Samuelson.

Samuelson's argument in support of this claim is incorrect. Under Equation (6) in [17, p. 94] he says: "For convenience, we may attach a number to each combination; this is assumed to be a continuous differentiable function." The representation of ordinal data by numerical scales is not a matter of convenience – it requires proof and it holds under specific assumptions. An ordinal scale cannot "be assumed" to be a continuous differentiable function: This assumption is self-contradictory – ordinal scales cannot be differentiable and cannot be assumed to be differentiable. The sentence (on the same page) of which Equation (9) is part, refers to the transformation F as "any monotonic scale" but in that sentence, F is "any monotonic *differentiable* scale." The monotonic *differentiable* scales are a proper subset of ordinal scale transformations – the set of ordinal scale transformations also contains non-differentiable monotone (increasing) transformations. Finally, the conditions for applying the chain rule, which is used in the partial differentiation on an indifference surface, do not hold for ordinal variables. For a detailed analysis of the ordinal utility error see Barzilai [3, §3.4].

4 Multiple Decision Makers

The subject of multiple decision makers is treated in game theory, welfare economics, and more narrowly in "Arrow's impossibility theorem" [1] (see Barzilai [4–6, and §§3.5–3.6 in 3] for details).

Game theory is founded on mathematical errors as indicated above. In particular, the underlying utility theory is self-contradictory, addition is applied where it is inapplicable, and game theory's characteristic function is ill-defined [5]. The concepts of von Neumann and Morgenstern's solution, imputations, and Shapley value, which depend on the characteristic function, are ill-defined as well.

The mathematical treatment in welfare economics and social choice is also flawed as it is founded on ordinal utility where addition and multiplication are not applicable and undefined sums appear throughout the literature (see e.g. Sen [18]).

Arrow's impossibility theorem is a negative result that deals with ordinal systems where addition and multiplication are not applicable. Scientific disciplines cannot be founded on ordinal systems nor on negative results and, in particular, the construction of preference scales cannot be founded on negative results.

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