Game Theory Foundational Errors – Part III

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Abstract

We have established in Barzilai [4–6] that *n*-person game theory is founded on errors. In this paper we show that two-person zero-sum game theory prescribes to the players "optimal" strategies that cannot be described as conservative or rational.

1 Introduction

Aumann tells us in the General Summary and Conclusions of his 1985 paper entitled "What is Game Theory Trying to Accomplish?" [2, p. 65] that "Game-theoretic solution concepts should be understood in terms of their applications, and should be judged by the quantity and quality of their applications." More recently, in their paper entitled "When All is Said and Done, How Should You Play and What Should You Expect?" Aumann and Dreze [3, p. 2] tell us that seventy-seven years after it was born in 1928, strategic game theory has not gotten beyond the optimal strategies which rational players should play according to von Neumann's minimax theorem of two-person zero-sum games; that when the game is not two-person zero-sum none of the equilibrium theories tell the players how to play; and that the "Harsanyi-Selten selection theory does choose a unique equilibrium, composed of a well-defined strategy for each player and having a well-defined expected outcome. But nobody – least of all Harsanyi and Selten themselves – would actually recommend using these strategies."

This implies that while the meaning of *n*-person "solutions" is in question, game theorists universally accept the minimax strategy as a reasonable (in fact, *the only)* solution for rational players in two-person zero-sum games. Consistent with this universal agreement is Aumann's characterization of the minimax theorem as a vital cornerstone of game theory in his survey of game theory [1, p. 6].

2 "How Should You Play" – According to Game Theory

To understand two-person zero-sum solutions in terms of their applications, consider the simplest game. The outcomes are given in the table

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where player 1 can choose between R1 and R2 (rows) and player 2 between C1 and C2 (columns) and *A, B, C, D* are the outcomes from player 1's perspective.

Suppose that the game describes competing businesses and that *A* represents a desirable outcome for player 1, *B* represents a slightly better outcome, *C* a slightly worse outcome and *D* represents a total ruin for player 1. For example, let *A, B, C, D* be the amounts in dollars gained by player 1 according to the table

$$\begin{bmatrix} 100{,}000 & 100{,}001 \\ 99{,}999 & -10{,}000{,}000 \end{bmatrix}.$$

It is easy to verify (for details see e.g. Hillier and Lieberman [7, Chapter 12]) that in this case the theory prescribes to the players the (pure) strategies (R1, C1). If player 1 chooses R1 he will gain 100,000 in the worst case while if he chooses R2 he will be ruined in the worst case by losing 10,000,000. By choosing strategy R1, which the theory prescribes to him, player 1 will not be ruined in the worst case. The prescription of considering the worst case for each choice and choosing the best of these worst cases (which is the minimax strategy for player 1), seems to represent a conservative choice in this case.

Now consider a small change in one of the table's entries:

$$\begin{bmatrix} 100{,}000 & 100{,}001 \\ 100{,}002 & -10{,}000{,}000 \end{bmatrix}.$$

For this revised game the theory prescribes probabilistic ("mixed") strategies to the players (the game has no saddle point). Specifically, according to two-person zero-sum game theory, player 1 "should" play strategy R2 with some non-zero probability $0 < p < 1$.

According to this prescription, player 1 should respond to a negligible change in the outcomes by adopting a strategy whereby, with a non-zero probability, he may be ruined. A businessman who displays such risk attitude is more likely to be considered an irresponsible gambler than a conservative, rational person. It follows that the answer to the question "How should you play?" according to this vital cornerstone of game theory is: *"risk everything!"*

This prescription, which applies to all zero-sum games of the form

$$\begin{bmatrix} L & L+\varepsilon \\ L+2\varepsilon & -M \end{bmatrix}$$

where $L,\ M,\ \varepsilon > 0$, is neither conservative nor rational when ε is small and M is much larger than L.

3 Analysis

These examples characterize the minimax solution of two-person zero-sum games in two ways: (i) All outcomes that do not represent the worst case for one of the players are ignored by the minimax solution. (ii) The minimum and maximum operations take into account only ordinal information and ignore the size of the differences of the entries in the tables.

4 Conclusions

Typically, in real-life applications the outcomes of conflict are not real numbers and a player does not play a game with the expectation of winning a real number. For example, among the possible outcomes of a game Luce and Raiffa [8, p. 57] list "Player 1 is killed and player 2 is maimed." If the outcome 'x' of a game stands for "player 2 loses his leg," the outcome 'y' for which the equation $x + y = 0$ holds is, at best, not clearly defined (player 2 "wins" player's 1 leg? player 1 loses his leg? etc.). Furthermore, in order for the concept of "zero-sum" to be defined, the operation of addition must be applicable. In Barzilai [4] we showed that this requires the transition from outcomes to preferences, i.e. utilities. However, utility functions are not unique, the concept of "zero-sum" is not properly defined for utilities, and there are additional difficulties with utility theory – see Barzilai [6].

In summary, the unavoidable conclusion from this analysis and Barzilai [4–6] is that as a branch of applied mathematics game theory fails Aumann's test.

References

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