On Ordinal, Cardinal, and Expected Utility

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Abstract

By formally defining the relevant mathematical spaces and models we show that the operations of addition and multiplication, and the concepts that depend on these operations, are not applicable on ordinal, cardinal, and expected utility. Furthermore, expected utility's scale construction rule is self-contradictory.

1 Introduction

Our purpose is to clarify some fundamental utility theoretical issues. While von Neumann and Morgenstern's utility axioms [7, p. 26] have attracted much attention, the framework in which they measure preference by constructing utility scales has been mostly overlooked and the applicability of mathematical operations on utility functions has been taken for granted in the literature of operations research and economic theory.

We define the relevant mathematical spaces and models and show that the operations of addition and multiplication, and the concepts that depend on these operations, are undefined and are not applicable on ordinal, cardinal, and expected utility functions.

2 Applicability of Operations: Mathematical Spaces

Mathematical spaces, e.g. vector or metric spaces, are sets of objects on which specific relations and operations (i.e. functions or mappings) are defined. They are distinguished by these relations and operations – unless explicitly specified, the objects are arbitrary.

Only those relations and operations that are defined in a given mathematical space are relevant and applicable when that space is considered – the application of undefined relations or operations is an error. For example, although the operations of addition and multiplication are defined in the *field* of real numbers, multiplication is undefined in the *group* of real numbers under addition; multiplication is not applicable in this group.

In all the spaces that follow the relation of equality (an equivalence relation) is assumed to be defined.

2.1 Ordinal Spaces

An ordinal space is a set A of objects equipped only with the relations of order and equality. Our interest is limited to the case of a complete order where for any $a, b \in A$ exactly one of $a < b$, $b < a$, or $a = b$ holds (the relation of order is irreflexive, antisymmetric, and transitive).

Since order and equality are not operations, i.e. single-valued functions, no operations are defined in ordinal spaces. Specifically, the operations of addition and multiplication (and their inverses – subtraction and division) are not applicable in ordinal spaces.

2.2 Vector Spaces

2.2.1 Groups and Fields

A **group** is a set G with a binary operation, denoted $a \circ b$, that satisfies the following axioms:

- The operation is *closed:* $c = a \circ b \in G$ for any $a, b \in G$.
- The operation is *associative:* $(a \circ b) \circ c = a \circ (b \circ c)$ for any $a, b, c \in G$.
- The group has an *identity:* there exists $e \in G$ such that $a \circ e = a$ for all $a \in G$.
- *Inverse elements:* for any $a \in G$, the equation $a \circ x = e$ has a unique solution x, the inverse of a, in G.

In addition, if $a \circ b = b \circ a$ for all $a, b \in G$, the group is *commutative.*

A **field** is a set F with two operations that satisfy the following axioms:

- The set F is a commutative group under the operation of *addition.*
- The set $F - \{0\}$, where zero is the additive identity, is a commutative group under the operation of *multiplication.*
- $a \times 0 = 0$ for any $a \in F$.
- For any $a, b, c \in F$ the distributive law $a \times (b + c) = (a \times b) + (a \times c)$ holds.

A **vector space** is a pair of sets (V, F) with associated operations as follows. F is a field and its elements are termed scalars. The elements of V are termed vectors and V is a commutative group under vector addition. For any scalars $j, k \in F$ and vectors $u, v \in V$ the scalar product $kv \in V$ is defined and satisfies, in the usual notation, $(j+k)v = jv + kv$, $k(u+v) = ku + kv$, $(jk)v = j(kv)$ and $1 \cdot v = v$.

2.3 Affine Spaces

An **affine space** is a triplet of sets (P, V, F) together with associated operations as follows (for equivalent definitions see Artzy [1] and Postnikov [8]). The pair (V, F) is a vector space. The elements of P are termed points and two functions are defined on points: a one-to-one and onto function $h : P \rightarrow V$ and a "difference" operation $\Delta : P^2 \rightarrow V$ that is defined by $\Delta(a, b) = h(a) - h(b)$.

The difference $\Delta : P^2 \rightarrow V$ is not a closed operation on P: although points and vectors can be identified through the one-to-one correspondence $h : P \rightarrow V$, the sets of points and vectors are equipped with different operations and the operations of addition and multiplication are not defined on points. If $\Delta(a, b) = v$, it is convenient to say that the difference between the points a and b is the vector v. Accordingly, we say that an affine space is equipped with the operations of (vector) addition and (scalar) multiplication *on point differences.*

The dimension of the affine space (P, V, F) is the dimension of the vector space V. In a one-dimensional affine space, for any pair of vectors $u, v \in V$ where $v \neq 0$ there exists a unique scalar $\alpha \in F$ so that $u = \alpha v$ and the set P is termed an affine straight line. In a one-dimensional vector space, the ratio $u / v = \alpha$ for $u, v \in V$, $v \neq 0$, means that $u = \alpha v$. Therefore, in an affine space, the expression $\Delta(a, b)/\Delta(c, d)$ for the points $a, b, c, d \in P$ where $\Delta(c, d) \neq 0$, is defined and is a scalar:

$$\frac{\Delta(a, b)}{\Delta(c, d)} \in F \tag{1}$$

if and only if the space is one-dimensional, i.e. a straight line. By definition, when the space is a straight line, $\Delta(a, b)/\Delta(c, d) = \alpha$ (where $a, b, c, d \in P$) means that $\Delta(a, b) = \alpha\Delta(c, d)$.

2.4 Ordered Affine Straight Lines

A field F is ordered if it contains a subset P such that if $a, b \in P$, then $a + b \in P$ and $a \times b \in P$, and for any $a \in F$ exactly one of $a = 0$, or $a \in P$, or $-a \in P$ holds. An ordered affine straight line is an affine straight line over an ordered field.

The relation of order, which is needed to indicate a direction on a straight line (for example, to indicate that an object is more preferable than another), is defined in an ordered affine straight line since it is an ordered one-dimensional space.

2.5 Expected Utility Spaces

Since expected utility axiom sets in the literature are not necessarily equivalent, we list here the main features of the von Neumann and Morgenstern's axioms [7 p. 26].

This space is equipped with two completely ordered sets: a set A of arbitrary objects, and a set I which is the subset of the ordered field of real numbers in the open interval (0, 1). No operations are defined on the set A, but a single ternary operation $e : I \times A \times A \rightarrow A$ is defined in this space. Additional assumptions impose constraints on the order and the operation but no other relations or operations are defined in an expected utility space.

3 Applicability of Operations: Models

Whether non-physical properties such as utility (i.e. preference) can be measured, and hence whether mathematical operations can be applied on scale values representing such properties, remained an open question when in 1940 a Committee appointed by the British Association for the Advancement of Science in 1932 "to consider and report upon the possibility of Quantitative Estimates of Sensory Events" published its Final Report (see Ferguson *et al.* [3]). An Interim Report, published in 1938, included "a statement arguing that sensation intensities are not measurable" as well as a statement arguing that sensation intensities are measurable. These opposing views were not reconciled in the 1940 Final Report (for additional details see Barzilai [2]).

For our purposes it is sufficient to note the following elements of the measurement framework: An empirical system E is a set of empirical objects together with operations, and possibly the relation of order, which characterize a property under measurement. A mathematical model M of the empirical system E is a set with operations that reflect the operations in E as well as the order in E when E is ordered. A scale s is a homomorphism from E into M, i.e. a mapping of the objects in E into the objects in M that reflects the structure of E into M. The purpose of modelling E by M is to enable the application of mathematical operations on the elements of the mathematical system M and mathematical operations in M are applicable if and only if they reflect empirical operations in E (see e.g. von Neumann and Morgenstern [7, §3.4]).

4 Ordinal Utility

An ordinal space, i.e. an ordered set, is not a Euclidean space. Since it is not a vector space, the elementary operations of addition and multiplication are not applicable in an ordinal space. Therefore, the operations and concepts of algebra and calculus are undefined in ordinal spaces. In particular, norms, metrics, derivatives, and convexity concepts are undefined and not applicable in an ordinal space. Therefore, ordinal utility functions are not differentiable and, conversely, differentiable scales cannot be ordinal and, since the partial derivatives of an ordinal utility function do not exist, the concept of marginal utility is undefined in an ordinal space.

Under the titles *Need for a theory consistently based upon ordinal utility* and *The ordinal character of utility* Hicks [5, Chapter I, §§4–5] proceeds "to undertake a purge, rejecting all concepts which are tainted by quantitative utility" [5, p. 19]. In essence, he claims that wherever utility appears in economic theory, and in particular in demand theory which employs partial differentiation, it can be replaced by ordinal utility. The notion of differentiable ordinal functions is untenable and has no parallel in mathematics and science: Thermodynamics is not and cannot be founded on ordinal temperature scales. Clearly, the concept of "slope," i.e. derivative, is undefined on an *ordinal* topographic map.

Hicks's untenable claim, which appears in current economic textbooks, was followed in Samuelson's *Foundations of Economic Analysis* [9, pp. 94–95] by a more technical, but incorrect, argument in support of this claim. This analysis is carried out in an unspecified space, which in fact is an ordinal space, and operations that are not applicable in this space are applied. For example, the chain rule of differentiation is applied where the conditions for applying this rule are not satisfied. Note also that the set of ordinal scale transformations contains *all* monotone increasing functions (if $u(x)$ is an ordinal utility function, so is $F(u(x))$ where F is *any* monotone increasing function) but Samuelson's chain rule argument applies only to the subset of *differentiable* ordinal scale transformations. (Consider for example the ordinal utility function $u(x_1, x_2)$ whose value is 1 when both variables are rational and 2 otherwise.) For additional details see Barzilai [2, §3.4].

5 Cardinal Utility

The concept of cardinal utility has no counterpart (e.g. cardinal time or cardinal temperature) in science. Saying that cardinal properties are those not preserved under all ordinal transformations amounts to saying that "cardinal" means "non-ordinal" which is not a proper definition. Some authors (e.g. Harsanyi [4, p. 40]) define cardinal utility functions as utility functions that are unique up to positive affine transformations (i.e. "interval" scales), but there is no mathematical definition of "cardinal space" in the literature and no proof that this scale-uniqueness type implies the applicability of the operations of addition and multiplication. In fact, it is easy to see that "interval" uniqueness *does not* imply the applicability of addition and multiplication.

6 Expected Utility

6.1 Inapplicability of Addition and Multiplication

Since various expected utility spaces differ only in the constraints they impose on the order relation and the expectation operation (they are equipped with *one ternary* operation), the operations of addition and multiplication (*two binary* operations) are not defined and are not applicable on expected utility scales.

6.2 The Expected Utility Rule is Self-Contradictory

The expected utility rule for lotteries, $u(l) = pu(a) + (1-p)u(b)$, imposes a constraint on the utility of the lottery $l = \{(p, a), (1-p, b)\}$ while no constraints are imposed on the utility of prizes. This rule is contradictory for prizes that are lottery tickets which the theory does not exclude.

7 Summary

It is not recognized in the literature (e.g. Hillier and Lieberman [6] and Harsanyi [4]) that the concepts of cardinal and expected utility are fundamentally flawed while the operations of algebra and calculus are not applicable on ordinal functions.

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