Ordinal Utility and Indifference Curves

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Abstract

The claim that modern economic theory can be founded on indifference curves of ordinal utility functions is based on errors. For the same reasons that the mathematical theory of thermodynamics cannot be founded on ordinal temperature scales, modern economic theory cannot be founded on ordinal data. Economic theory should be corrected using utility scales that enable the "powerful weapons" of algebra and calculus.

1 Introduction

In his *Manual of Political Economy*, Pareto claims that "the entire theory of economic equilibrium is independent of the notions of (economic) *utility"* [7, p. 393]. More precisely, it is claimed that ordinal utility scales are sufficient to carry out Pareto's development of economic equilibrium. This claim is surprising considering that Pareto's *Manual* is founded on the notions of differentiable utility scales (by different names such as "ophelimity" and "tastes"). This claim is also surprising because a parallel claim stating that *ordinal temperature scales are sufficient to carry out partial differentiation in thermodynamics* is obviously false. It is even more surprising that this false claim has escaped notice for so long and is repeated in current economic literature. The purpose of this paper is to illuminate the source of the error, highlight some of its implications, and point out the necessary correction.

Relying on Pareto's error, Hicks [6, p. 18] states that "The quantitative concept of utility is not necessary in order to explain market phenomena." With the goal of establishing a *Logical foundation of deductive economics* – having identified the *Need for a theory consistently based upon ordinal utility* – (see the titles of Chapter I's sections in *Value and Capital* [6]) he proceeds "to undertake a purge, rejecting all concepts which are tainted by quantitative utility" [6, p. 19]. In essence, Hicks claims that wherever utility appears in economic theory, and in particular in Pareto's theory which employs partial differentiation, it can be replaced by ordinal utility (see also the title *The ordinal character of utility* [6, Chapter I, §4]).

Neither Pareto, who did not act on his claim, nor Hicks, who did proceed to purge "quantitative utility" from economic theory, attempted to provide rigorous mathematical justification for this claim and it seems that authors who repeat this claim rely on an incorrect argument in Samuelson's *Foundations of Economic Analysis* [8, pp. 94–95].

2 Ordinal Utility

The relevant issues concerning ordinal utility scales are as follows (see Barzilai [4 and 5] for details). An ordinal empirical system E is a set of empirical objects together with the relation of order, which characterize a property under measurement. A mathematical model M of the empirical system E is an ordered set where the order in M reflects the order in E. A scale s is a homomorphism from E into M, i.e. a mapping of the objects in E into the objects in M that reflects the order of E into M. In general, the purpose of modelling E by M is to enable the application of mathematical operations on the elements of the mathematical system M and operations that are not defined in E are not applicable in M – see *The Principle of Reflection* in Barzilai [4, §3.2]. In the case of ordinal systems the mathematical image M of the empirical system E is equipped only with order and the operations of addition and multiplication are not applicable in M. In other words, since, by definition, in ordinal systems only order is defined (explicitly – neither addition nor multiplication is defined), addition and multiplication are not applicable on ordinal scale values and it follows that the operation of differentiation is not applicable on ordinal scale values because differentiation requires that the operations of addition and multiplication be applicable.

In summary, if $u(x_1, \ldots, x_n)$ is an ordinal utility function it cannot be differentiated and conversely, a utility function that satisfies a differential condition cannot be an ordinal utility scale.

3 Optimality Conditions on Indifference Surfaces

In [6, p. 23] Hicks says that "Pure economics has a remarkable way of producing rabbits out of a hat" and that "It is fascinating to try to discover how the rabbits got in; for those of us who do not believe in magic must be convinced that they got in somehow." The following is treated with only that minimal degree of rigor which is necessary to discover how this observation applies to the use of, supposedly ordinal, utility functions in the standard derivation of elementary equilibrium conditions. (A greater degree of rigor is necessary if other errors are to be avoided.)

Consider the problem of maximizing a utility function $u(x_1, \ldots, x_n)$ subject to a constraint of the form $g(x_1, \ldots, x_n) = b$ where the variables $x_1, \ldots, x_n$ represent quantities of goods. Differentiating the Lagrangean $L = u - \lambda(g - b)$ we have

$$\frac{\partial u}{\partial x_i} - \lambda \frac{\partial g}{\partial x_i} = 0 \text{ for } i = 1, \ldots, n$$

which implies $\frac{\partial u}{\partial x_i} \div \frac{\partial g}{\partial x_i} = \lambda = \frac{\partial u}{\partial x_j} \div \frac{\partial g}{\partial x_j}$ for all i, j, and therefore

$$\frac{\partial u}{\partial x_j} \div \frac{\partial u}{\partial x_i} = \frac{\partial g}{\partial x_j} \div \frac{\partial g}{\partial x_i}. \tag{1}$$

Equation (1) is a tangency condition because, in common notation,

$$\frac{\partial x_i}{\partial x_j} = -\left(\frac{\partial f}{\partial x_j} \div \frac{\partial f}{\partial x_i}\right) \tag{2}$$

holds on a surface where a function $f(x_1, \ldots, x_n)$ is constant. Since applying this notation to Equation (1) yields

$$\frac{\partial x_i}{\partial x_j} = \frac{\partial x_i}{\partial x_j},$$

it is preferable to use the explicit notation

$$\left.\frac{\partial x_i}{\partial x_j}\right|_x^u$$

to indicate that the differentiation is performed on an indifference surface of the function u at the point x. This derivative depends on the function u as well as the point x; the function u is not "eliminated" in this expression. In general, at an arbitrary point x we expect

$$\left.\frac{\partial x_i}{\partial x_j}\right|_x^u \neq \left.\frac{\partial x_i}{\partial x_j}\right|_x^g$$

but at the solution point x^* Equation (1) implies

$$\left.\frac{\partial x_i}{\partial x_j}\right|_{x^*}^u = \left.\frac{\partial x_i}{\partial x_j}\right|_{x^*}^g \quad \text{for all } i, j \tag{3}$$

which, together with the constraint $g(x_1, \ldots, x_n) = b$, is a system of equations for the n unknowns $x^* = (x_1^*, \ldots, x_n^*)$.

In the special case of a budget constraint $p_1 x_1 + \ldots + p_n x_n = b$ where p_i is the price of good i,

$$-\frac{\partial x_i}{\partial x_j}\bigg|_x^g = \frac{\partial g}{\partial x_j} \div \frac{\partial g}{\partial x_i} = \frac{p_j}{p_i}$$

and the solution satisfies

$$p_1 x_1^* + \ldots + p_n x_n^* = b \text{ and } -\frac{\partial x_i}{\partial x_j}\bigg|_{(x_1^*, \ldots, x_n^*)}^u = \frac{p_j}{p_i} \text{ for all } i, j. \tag{4}$$

When the number of variables is greater than two, this system of equations cannot be solved by the method of indifference curves, i.e. by using two-dimensional diagrams, because the left hand sides of the equations in (4) depend on all the n unknowns. For example, we can construct a family of indifference curves in the (x_1, x_2) plane where the variables $x_3, \ldots, x_n$ are fixed, but $x_3, \ldots, x_n$ must be fixed at the unknown solution values $x_3^*, \ldots, x_n^*$. To emphasize, with each fixed value of the variables $x_3, \ldots, x_n$ is associated a family of (x_1, x_2) indifference curves. To solve for x_1^*, x_2^* by the method of indifference curves, it is necessary to construct the specific family of indifference curves that corresponds to the solution values $x_3^*, \ldots, x_n^*$, but these values are not known. Noting again that the utility function u is not eliminated in Equation (4) and that this equation was derived using the operation of differentiation which is not applicable on ordinal utility functions, we conclude that Hicks's "Generalization to the case of many goods" [6, §9] has no basis.

Returning to Equation (2), we note that $f(x_1, \ldots, x_n)$ and $F(f(x_1, \ldots, x_n))$ have the same indifference surfaces (but with different derivatives) and, by the chain rule, if F and $f(x_1, \ldots, x_n)$ are both differentiable then

$$\frac{\partial x_i}{\partial x_j}\bigg|_x^{F(f)} = \frac{\partial x_i}{\partial x_j}\bigg|_x^f \tag{5}$$

so that this partial derivative is independent of F. However, since both F and f are assumed to be differentiable, Equation (5) does not imply that f is ordinal.

4 Pareto's Claim

In the Appendix to his *Manual of Political Economy* [7, pp. 392–394] Pareto considers the indifference surfaces of the utility $I = \Psi(x, y, z, \ldots)$ of the goods $x, y, z, \ldots$. Taking for granted the applicability of the operation of differentiation, if $I = F(\Psi)$ "is differentiated with I taken as a constant," Pareto obtains the equation (numbered (8) in his Appendix) $0 = \Psi_x dx + \Psi_y dy + \Psi_z dz + \ldots$ independently of F. This equation is followed by the statement that “An equation equivalent to the last mentioned could be obtained directly from observation.” Pareto then says that the latter equation (numbered (9) in his Appendix), $0 = q_x dx + q_y dy + q_z dz + \ldots$, “contains nothing which corresponds to ophelimity, or to the indices of ophelimity” (where he uses the term ophelimity for utility) and concludes that “the entire theory of economic equilibrium is independent of the notions of (economic) *utility”* [7, p. 393].

This conclusion has no basis: “direct observation” does not constitute mathematical proof; Pareto does not define the variables $q_x, q_y, q_z, \ldots$; and it is not clear what it is which he directly observes. To the contrary, if Pareto’s equation

$$0 = q_x dx + q_y dy + q_z dz + \ldots$$

contains nothing which corresponds to utility, it cannot be equivalent to his equation

$$0 = \Psi_x dx + \Psi_y dy + \Psi_z dz + \ldots$$

which characterizes utility indifference surfaces. As pointed out in §2, since Ψ satisfies a differential condition it cannot be an ordinal utility scale.

5 Samuelson’s Explanation

Samuelson defines an ordinal utility scale $\varphi(x_1, \ldots, x_n)$ in Equations (6)-(8) of [8, p. 94] and states, correctly, that any function $U = F(\varphi)$ where $F'(\varphi) > 0$ reflects the same order. However, this observation does not imply that φ is ordinal. To the contrary, since this observation is based on differentiating both F and φ, it is only valid if φ is differentiable in which case it cannot be ordinal.

The paragraph that follows this observation in [8, p. 94] consists of one sentence: “To summarize, our ordinal preference field may be written [here Samuelson repeats his Equation 9 as Equation 10] where φ is any one cardinal index of utility.” Recalling Hicks’s comment that “It is fascinating to try to discover how the rabbits got in,” this sentence is remarkable, for “those of us who do not believe in magic” will note that the *ordinal* utility at the beginning of the sentence has metamorphosed into *cardinal* utility at the sentence’s end. Note that Samuelson does not define the concept of “cardinal” utility, nor does it appear to be defined elsewhere in the literature.

The concepts of tangents, partial derivatives, and differentials that follow on the next page (Samuelson [8, p. 95]) are applicable only if the utility scales in question are differentiable in which case they cannot be ordinal. Additional analysis of the rest of Samuelson’s explanation is not necessary, except that it should be noted that the *marginal utilities* that appear in Equation (25) that follows on [8, p. 98] are partial derivatives of a utility function. If the derivatives of this utility function, i.e. the marginal utilities, exist it cannot be ordinal.

Finally, Samuelson’s use of preference and utility as synonyms is consistent with the discussion in Barzilai [3, §2.1].

6 Implications

Define an *ordinal utility* function of two variables by $u(x, y) = xy$ if x or y is a rational number, and $u(x, y) = x^3 y$ otherwise. Under the budget constraint $p_1 x + p_2 y = b$ the tangency condition

$$-\frac{\partial y}{\partial x}\bigg|^{u} = \frac{p_1}{p_2}$$

does not hold because (regardless of how the "or" in the definition of $u(x, y)$ is interpreted) the left hand side of this equation is undefined – the derivative does not exist.

This counter-example shows that ordinal utility scales are not sufficient for the derivation of the standard equilibrium conditions of consumer demand theory. Other parts of modern economic theory will have to be discarded as well if the theory is to be purged of "quantitative utility." Rather than discarding the advances of the last century, economic theory should be corrected and re-founded on strong utility scales that enable the operations of algebra and calculus.

7 Conclusions

The claim that modern economic theory can be founded on indifference curves of ordinal utility functions is based on errors. Current economic theory contains errors that should be corrected using utility scales that enable the operations of algebra and calculus, i.e. strong affine scales. The mathematical structure of such scales is identical to the structure of similar scales that represent physical variables such as potential energy, position, and time, so that the difficulties are conceptual rather than technical. The need to correct the mathematical foundations of economic theory (as well as what is now called measurement theory) extends to all the social sciences, and game theory requires major corrections too – see Barzilai [1–5].

References

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