And the Mathematics is Incorrect

The request for mathematical-decision-theory guidance in SIAM News December 2015's "The Models are Incomplete, the Intuitions are Unreliable," is commendable. The stakes are high, financially (billion-dollar decisions) and otherwise, and the SIAM community is well-equipped to advance the state of this branch of applied mathematics.

Whether and which mathematical operations are applicable on scale values for physical and non-physical, i.e. psychological, variables is a foundations-of-science question. This remained an open question when a Committee appointed by the British Association for the Advancement of Science in 1932 "to consider and report upon the possibility of Quantitative Estimates of Sensory Events" published its Final Report in 1940 (Ferguson et al. [4]). An Interim Report, published in 1938, included "a statement arguing that sensation intensities are not measurable" as well as a statement arguing that sensation intensities are measurable. These opposing views were not reconciled in the 1940 Final Report and mathematical operations have been applied since then without foundation and where they are not applicable in decision theory, mathematical economics, and other social disciplines (Barzilai [1 and 2]).

The most glaring of these errors have been committed in mathematical economics where the operation of differentiation is applied on ordinal data in a space where addition and multiplication are inapplicable. Inapplicable mathematical operations are also applied in decision theory which is concerned with non-physical variables such as preference, utility, or value. These errors affect the practice of decision making and what students of microeconomics and decision analysis are taught. An outline of the source of these errors and how they may be corrected follows.

Measurement is the process of scale construction for the variables of science and the social disciplines, and measurement scales are those homomorphisms that reflect the specific empirical operations and relations which characterize a given property to corresponding operations and relations in a mathematical model. The purpose of modelling an empirical system by a mathematical one is to enable the application of mathematical operations on the elements of the mathematical system: As Campbell says, "the object of measurement is to enable the powerful weapon of mathematical analysis to be applied to the subject matter of science" [3, pp. 267–268]. This measurement framework, which is due to Helmholtz [5], is the only basis for the introduction of mathematics into any discipline. It has been universally accepted since 1887 – see for example Campbell [3, Ch. X, 1920], von Neumann and Morgenstern [9, §3.4, 1944], Krantz et al. [7, pp. 8–9 and Chs. 2–4, 1971], Roberts [10, §2.1, 1979], and Barzilai [1, §3.2, 2010].

Given empirical operations on a set of objects, if there exists a scale (i.e. a homomorphism) that reflects these operations to corresponding mathematical ones, then the reflected mathematical operations, and only these operations, are applicable on the scale's values. In other words, the only operations that are *enabled* by a homomorphism from an empirical system to a mathematical model are those which are images of corresponding empirical operations, and these are the only mathematical operations that are *applicable* on scale values. In particular, if the only empirical relation is order and a scale reflects this order into an ordered set (typically the ordered real numbers), then the operations of addition and multiplication are not applicable on scale values. In this case the empirical system is ordinal and the only relation on scale values is order.

In a 1915 paper [12] Slutsky addresses the problem of applicability of mathematical methods to economic theory in the context of the theory of value. His treatment is founded on fundamental errors: Since order is the only relation in his definition of utility, no operations are applicable on his utility functions, his utility space is ordinal, his second derivatives of the utility function are undefined, and the discussion that follows has no foundation.

Hicks amplifies Slutsky's errors in his 1939 *Value and Capital* [6, p. 19]. But while he states that his theory is essentially Slutsky's, Slutsky *ignores the empirical system* of which the mathematical one is a model, whereas Hicks *purges the mathematical system* by undertaking "a purge, rejecting all concepts which are tainted by quantitative utility." This claimed purge of all concepts which are tainted by quantitative utility is contradicted by an (incorrect) analysis of the ratio of marginal utilities, i.e., by applying the mathematical operation of division on quantitative partial derivatives of utility functions (in an ordinal space).

Following Slutsky and Hicks, Samuelson [11, pp. 94–95] adds to their errors. Applying the tools of differential calculus where the assumptions for their applicability are not satisfied, he purportedly proves that marginal utility ratios can be derived from ordinal data. For a detailed analysis of the ordinal utility claim see Barzilai [1, §3.4]. More than a hundred years after the publication of Slutsky's paper, these errors are yet to be corrected in microeconomics textbooks, including advanced ones (e.g. Mas-Colell et al. [8]).

Operations that are not defined in a given mathematical space are inapplicable in that space, yet the application of inapplicable operations is a common decision-theory error. Correctly identifying the mathematical space in which addition and multiplication are applicable on the variables of decision theory and mathematical economics is necessary if the application of inapplicable operations in these disciplines is to be avoided. This space is the vector space that underlies a one-dimensional affine space (see Barzilai [1, §3.7]).

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Jonathan Barzilai, D.Sc.
Professor
Dalhousie University