Economic Theory Errors

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Until recently it has been assumed that the problem of constructing preference scales on which mathematical operations can be performed was solved by von Neumann and Morgenstern's utility theory. This is an important problem because until the publication of von Neumann and Morgenstern's *Theory of Games and Economic Behavior* in 1944, the possibility of measurement of non-physical variables such as preference had been an open question and also because preference measurement underpins economic theory, the theory of games, and decision theory. Recent research[1] has revealed errors at the foundations of economic theory, game theory, and other disciplines including the inapplicability of the operations of addition and multiplication on utility scale values. The mathematical foundations of economic theory have been reconstructed but additional corrections are required.

Can Psychological variables be measured?

The construction of the mathematical foundations of any scientific discipline requires the identification of the conditions that must be satisfied in order to enable the application of the mathematical operations of linear algebra and calculus. In addition, the mathematical foundations of social science disciplines, including economic theory, require the application of mathematical operations to *non-physical variables,* i.e, to variables that describe psychological or subjective properties such as *utility* or *preference.*

Whether psychological properties can be measured, and hence whether mathematical operations can be applied to psychological variables, remained an open question when in 1940 a Committee appointed by the British Association for the Advancement of Science in 1932 "to consider and report upon the possibility of Quantitative Estimates of Sensory Events" published its Final Report. An Interim Report, published in 1938, included "a statement arguing that sensation intensities are not measurable" as well as a statement arguing that sensation intensities are measurable. These opposing views were not reconciled in the Final Report.

The position that psychological variables cannot be measured is summarized by J. Guild in the Final Report in the context of measurement of *sensation* as follows:

1 This is a summary of J. Barzilai, "Preference Function Modeling: The Mathematical Foundations of Decision Theory," pp. 1–37, to appear in *Trends in MCDA,* José Figueira, Salvatore Greco, Matthias Ehrgott (Eds.). A pre-print is posted at www.ScientificMetrics.com

> I submit that any law purporting to express a quantitative relation between sensation intensity and stimulus intensity is not merely false but is in fact meaningless unless and until a meaning can be given to the concept of addition as applied to sensation. No such meaning has ever been defined. When we say that one length is twice another or one mass is twice another we know what is meant: we know that certain practical operations have been defined for the addition of lengths or masses, and it is in terms of these operations, and in no other terms whatever, that we are able to interpret a numerical relation between lengths and masses. But if we say that one sensation intensity is twice another nobody knows what the statement, if true, would imply.

The Mathematical Modelling Framework

To re-state Guild's position in current terminology the following is needed. By an empirical system E we mean a set of empirical *objects* together with *operations* (i.e. functions) and possibly the relation of *order* which characterize the property under measurement. A mathematical model M of the empirical system E is a set with operations that reflect the empirical operations in E as well as the order in E when E is ordered. A scale s is a mapping of the objects in E into the objects in M that reflects the structure of E into M (in technical terms, a scale is a homomorphism from E into M).

The purpose of modelling E by M is to enable the application of mathematical operations on the elements of the mathematical system M: "the object of measurement is to enable the powerful weapon of mathematical analysis to be applied to the subject matter of science" (Campbell, 1920).

The framework of mathematical modelling is essential. To enable the application of mathematical operations in a given empirical system, the empirical objects are mapped to mathematical objects on which these operations are performed. In mathematical terms, these mappings are functions from the set of empirical objects to the set of mathematical objects (typically the real numbers for reasons that are explained by the reconstructed theory). Given two sets, a large number of mappings from one to the other can be constructed, most of which are not related to the characterization of the property under measurement: A given property must be characterized by empirical operations which are specific to this property and these property-specific empirical operations are then reflected to corresponding operations in the mathematical model. Measurement scales are those mappings that reflect the specific empirical operations which characterize the given property to corresponding operations in the mathematical model. Empirical addition can easily be described for variables such as *mass* and *length* and it has been implicitly assumed that the structure of psychological scales is similar to the structure of *mass* and *length* scales.

In terms of this universally accepted fundamental framework, Guild states that for psychological variables it is not possible to construct a scale that reflects the empirical operation of addition because such an empirical (or "practical") addition operation has not been defined; if the empirical operation does not exist, its mathematical reflection does not exist as well.

Von Neumann and Morgenstern's Game Theory

In *Theory of Games and Economic Behavior* von Neumann and Morgenstern proposed game theory as "the proper instrument with which to develop a theory of economic behavior." Applying mathematical methods to economic theory requires the application of the mathematical operations which these methods employ to economic variables including *utility* or *preference* which, in turn, requires addressing the problem of psychological measurement since preference is a not a physical property of empirical objects. In particular, if the operations of addition and multiplication, which are elementary mathematical tools, are not applicable, very limited results can be attained.

Since establishing the applicability of addition and multiplication is a prerequisite for a discussion of the mathematical foundations of economic theory, von Neumann and Morgenstern needed to construct a mathematical model for *preference* measurement in which addition and multiplication are applicable.

Von Neumann and Morgenstern's Error

Measurement scales for *mass* or *length* are unique up to a multiplicative constant so that the scale $t = q \times s$ for $q > 0$ is equivalent to the scale *s* for these variables. Since scales for physical variables such as *time* and *potential energy* are unique up to an additive and a multiplicative constant ($t = p + q \times s$), the structure of psychological scales is not necessarily similar to the structure of *mass* and *length* scales. Motivated by this uniqueness argument, von Neumann and Morgenstern constructed a mathematical model for preference measurement, based on an empirical operation that mimics the "center of gravity" operation, where the scales that satisfy their utility axioms are unique up to an additive and a multiplicative constant.

Until recently it has not been realized that this construction does not solve the problem that von Neumann and Morgenstern needed to solve. What is needed is a construction of preference scales where the operations of addition and multiplication are applicable in the mathematical system *M* which is the range of each scale, i.e. operations such as $s(a) = s(b) + s(c)$ where the sum of two elements of *M* is another element of *M*. In scale transformations of the form $t = p + q \times s$ the operations are not performed in *M* but in the set *S* of all scales where one scale in *S* is transformed into another element of *S*. Addition and multiplication in scale transformations of the form $t = p + q \times s$, which characterize scale uniqueness, do not imply that addition and multiplication are applicable on scale values in *M* and it follows that the problem of applicability of addition and multiplication on scale values was not solved by von Neumann and Morgenstern and consequently, until recently, there has been no basis for the application of these operations in economic theory.

This has led to application of mathematical operations in an incorrect form or where they are not applicable and to additional errors in game theory, economic theory, and other social sciences. The use of utility sums in game theory and economics is an error: The correct model for *position*, an elementary variable of geometry and physics, is that of a one-dimensional affine space, i.e. a straight line with unmarked *zero* and *one*, as there is neither an absolute zero nor absolute one in this space (the space is a homogeneous field). In an affine space the sum of points is undefined. For example, since *potential energy* and *time* have no absolute zero or one, they are modelled by an affine straight line and the sum of potential energies $e_1 + e_2$ or times $t_1 + t_2$ is undefined. To emphasize, even on a single time scale with one and the same unit, $t_1 + t_2$ is undefined. Point differences in an affine space form a vector space and the sum of potential energy *differences* $\Delta e_1 + \Delta e_2$ and time *differences* $\Delta t_1 + \Delta t_2$ are defined. The implication for game theory and economics is that where addition and multiplication are enabled, i.e. for affine utility scale values, utility sums are undefined. This is the case not just in welfare economics which deals with utility sums of different persons but also in the case of a utility scale of a single person: the utility sum $u(a) + u(b)$ is undefined.

Another fundamental error in economic theory is the notion that ordinal utility scales are sufficient to carry out differentiation in economic theory. The operation of differentiation is not applicable on ordinal scales because addition and multiplication are not applicable on such scales. It is inconceivable that anyone would claim that ordinal temperature measurement is a sufficient foundation for the operation of differentiation in thermodynamics yet, in his *Manual of Political Economy*, Pareto claims that "the entire theory of economic equilibrium is independent of the notions of (economic) *utility"* by which is meant that ordinal utility scales are sufficient to carry out the development of economic equilibrium theory where marginal utility – the partial derivative of a utility function – plays a central role. Pareto's claim has been amplified by other economists and appears throughout the literature of modern economic theory. An ordinal utility function cannot be differentiated and, conversely, a utility function that satisfies a differential condition cannot be an ordinal utility scale.

In game theory, the undefined sum $v(S) + v(T)$ where $v(S)$ and $v(T)$ are the values of coalitions S and T, appears in the definition of von Neumann and Morgenstern's characteristic function of a game, a central concept of the theory. Similarly, the sum of imputations, which are utilities, is undefined and throughout the literature of game theory, the treatment of the division of the "payoff" among the players in a coalition has no foundation. The characteristic function is ill-defined for another reason as well: The value of an object is not a physical property of the object and the definition of *value* requires specifying both *what* is being valued and *whose* values are being measured, but *whose* values are being reflected by the characteristic function is not specifed in the theory. All game theory concepts that depend on values where it is not specified whose values are being measured are ill-defined including the concept of imputations, von Neumann and Morgenstern's solution of a game, and Shapley's Value in all its variants and generalizations. Moreover, since the current definition of an *n*-person game employs the ill-defined concept of the characteristic function, this definition of a game has no foundation.

The application of mathematical operations such as addition and multiplication requires the mathematical modelling of economic systems by corresponding mathematical systems. Since the property under measurement is an integral part of the mathematical modelling framework and money is not a property of objects, preference measurement is the only way to introduce the real numbers and operations on them to economics and game theory. It follows that it is not possible to escape the need to construct preference functions by assuming that payoffs are in money units and that each player has a utility function which is linear in terms of money.

These are not the only errors at the foundations of game theory. Aumann and Dreze write under the title "When All is Said and Done, How Should You Play and What Should You Expect?" that seventy-seven years after it was born in 1928, strategic game theory has not gotten beyond the optimal strategies which rational players should play according to von Neumann's minimax theorem of two-person zero-sum games; that when the game is not two-person zero-sum none of the equilibrium theories tell the players how to play; and that the "Harsanyi-Selten selection theory does choose a unique equilibrium, composed of a well-defined strategy for each player and having a well-defined expected outcome. But nobody – least of all Harsanyi and Selten themselves – would actually recommend using these strategies." This indicates that while the meaning of *n*-person games' solutions is in question, game theorists universally accept the minimax strategy as a reasonable – in fact *the only* – solution for rational players in two-person zero-sum games. However, the minimax solution of two-person zero-sum game theory, which Aumann considers a vital cornerstone of game theory, prescribes to the players "optimal" strategies that cannot be described as conservative or rational; "the" value of two-person zero-sum game theory is not unique and consequently is ill-defined; and the minimax solution divorces choice probabilities from choice consequences which is a fundamental error that indicates that this problem is formulated incorrectly.

Utility theory, which underpins economic theory as well as the theory of games and decision theory, cannot serve as a foundation for mathematical methods in any scientific discipline. The operations of addition and multiplication are not applicable on scale values in any version of utility theory and, in addition to other shortcomings, although von Neumann and Morgenstern's utility axioms are consistent in the abstract, the interpretation of the empirical utility operation in terms of lotteries and prizes creates an intrinsic contradiction: The theory permits lotteries that are prizes and has a rule for assigning values to prizes and a different, conflicting, rule for assigning values to lotteries. For a prize which is a lottery ticket, the conflicting rules are contradictory.

In summary, the fundamental issue of applicability of the operations of addition and multiplication to scale values was not resolved by von Neumann and Morgenstern's utility theory and the mathematical foundations of economic theory and other social sciences need to be corrected to account for the conditions that must be satisfied for the mathematical operations of linear algebra and calculus to be applicable.