Demand, Barter, and Exchange

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Abstract
This paper presents the mathematical formulation of an economic exchange transaction.

Keywords
Demand theory, utility theory, barter, exchange.

1 Introduction

In Barzilai [1 and 2] we pointed out the need to correct demand theory errors. This paper presents the mathematical formulation of an exchange transaction with or without money.

2 An Unconstrained Maximization Formulation

Consider the problem of maximizing the utility function $u(q_1 + x_1, \ldots, q_n + x_n)$ with respect to the quantity changes $x_1, \ldots, x_n$ in an exchange transaction involving n commodities. This formulation, where the utility function and the initial quantities $q_1, \ldots, q_n$ are given and the variables $x_1, \ldots, x_n$ are bounded implicitly by the initial quantities, applies when money is one of the variables (goods are purchased or sold) as well as in the case of a barter transaction where money is not one of the variables. When money is one of the commodities, it must be one of the utility function's arguments, which is not the case for the utility functions that define Marshallian and Hicksian demand functions.

If the utility function is differentiable and its maximum is attained at a point in the interior of its domain, the quantity changes $x_1, \ldots, x_n$ are a solution of the first order necessary conditions

$$\frac{\partial}{\partial x_i} u(q_1 + x_1, \ldots, q_n + x_n) = 0 \text{ for } i = 1, \ldots, n. \tag{1}$$

For example, consider the utility function

$$u(x, y) = 100 - [(q_x + x - 10)^2 + (q_y + y - 10)^2]$$

where $0 \le q_x + x, q_y + y \le 10$. For the initial quantities $(q_x, q_y) = (2, 3)$, the problem is to maximize $u(x, y) = 100 - [(2 + x - 10)^2 + (3 + y - 10)^2]$ and the solution, which satisfies the equations $2 + x - 10 = 0$ and $3 + y - 10 = 0$, does depend on the initial quantities. Note that the exchange rates in this case, $\frac{\partial y}{\partial x} = -\frac{\partial u}{\partial x} \div \frac{\partial u}{\partial y} = -\frac{q_x + x - 10}{q_y + y - 10}$, depend on the initial quantities and are not constant.

3 The Case of Constant Exchange Rates

For a linear utility function $u(x) = \sum_{i=1}^{n} \alpha_i x_i$ the exchange rates $\frac{\partial u}{\partial x_i} \div \frac{\partial u}{\partial x_j} = \frac{\alpha_i}{\alpha_j}$ are constants and the indifference surfaces are hyperplanes (in the two-dimensional case the indifference surfaces are curves and these curves are straight lines). Conversely, if the exchange rates are constants in an open domain of a smooth $u(x)$ then it is essentially linear as it must be of the form

$$u(x) = f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \tag{2}$$

for some differentiable function of one variable $f(x)$, in which case the indifference surfaces are hyperplanes (for a proof see Theorem 2 in Barzilai [6]). For example, since the logarithm of a Cobb-Douglas function is a linear function, it is essentially linear, and its indifference surfaces are hyperplanes.

In addition, since the indifference surfaces are hyperplanes if and only if $u(x)$ is essentially linear, the exchange rates cannot be constant for a utility function that is not essentially linear. This implies that in the special case of fixed prices, e.g. in consumer-demand theory, where the exchange rates are constants, once all the variables are taken into account (in contrast with the Marshallian and Hicksian formulations which ignore the utility of money), the indifference surfaces are hyperplanes.

4 Conclusion

There is no mathematical formulation of the basic economic transaction of exchange with or without money in the literature. Microeconomic theory is deficient for this and additional reasons – see Barzilai [1–5].

References

[1] Jonathan Barzilai, *A Unified Consumer Demand Theory,* 2016. Posted at http://scientificmetrics.com/publications.html

[2] Jonathan Barzilai, *Slutsky's Mathematical Economics,* 2016. Posted at http://scientificmetrics.com/publications.html

[3] Jonathan Barzilai, *An Open Letter to the President of the American Economic Association,* September 2014. Posted at http://scientificmetrics.com/publications.html

[4] Jonathan Barzilai, "Inapplicable Operations on Ordinal, Cardinal, and Expected Utility," *Real-World Economic Review*, No. 63, pp. 98–103, 25 March 2013.

[5] Jonathan Barzilai, "Preference Function Modeling: The Mathematical Foundations of Decision Theory," in *Trends in Multiple Criteria Decision Analysis,* Matthias Ehrgott, José Rui Figueira, Salvatore Greco (Eds.), Springer, pp. 57–86, 2010.

[6] Jonathan Barzilai, "On the Decomposition of Value Functions," *Operations Research Letters,* Vol. 22, Nos. 4–5, pp. 159–170, 1998.