On Microeconomics Errors
and
Ordinal Group Decision Making

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Abstract

The notion of the 'mathematical impossibility' of group decision making is founded on errors in microeconomics, game theory, and related disciplines. We highlight mathematical errors that have been committed by von Neumann and Morgenstern, Pareto, Hicks, Samuelson, and Debreu which have been propagated throughout the social sciences.

1 Applicability of Mathematical Operations

1.1 Mathematical Spaces

Mathematical spaces, e.g. vector or metric spaces, are sets of objects on which specific relations and operations (i.e. functions or mappings) are defined. They are distinguished by these relations and operations – unless explicitly specified, the objects are arbitrary.

Only those relations and operations that are defined in a given mathematical space are relevant and applicable when that space is considered: The application of undefined relations or operations is an error. For example, although the operations of addition and multiplication are defined in the *field* of real numbers, multiplication is undefined in the *group* of real numbers under addition and, therefore, multiplication is not applicable in this group.

1.2 Models

Whether non-physical properties such as utility (i.e. preference) can be measured, and hence whether mathematical operations can be applied on scale values representing such properties, remained an open question in 1940 when a Committee appointed by the British Association for the Advancement of Science in 1932 "to consider and report upon the possibility of Quantitative Estimates of Sensory Events" published its Final Report (see Ferguson *et al.* [9]). An Interim Report published in 1938 included "a statement arguing that sensation intensities are not measurable" as well as a statement arguing that sensation intensities are measurable. These opposing views were not reconciled in the 1940 Final Report (for additional details see Barzilai [5]).

For our purposes it is sufficient to note the following elements of the measurement framework: An empirical system E is a set of empirical objects together with operations, and possibly the relation of order, which characterize a property under measurement. A mathematical model M of the empirical system E is a set with operations that reflect the operations in E as well as the order in E when E is ordered. A scale s is a homomorphism from E into M, i.e. a mapping of the objects in E into the objects in M that reflects the structure of E into M. The purpose of modelling E by M is to enable the application of mathematical operations on the elements of the mathematical system M, and mathematical operations in M are applicable if and only if they reflect empirical operations in E (see e.g. von Neumann and Morgenstern [16, §3.4]).

2 Ordinal Spaces

An ordinal space is a set A of objects equipped only with the relations of order and equality. The relation of order is irreflexive, antisymmetric, and transitive – our interest is limited to the case of a complete order where for any $a, b \in A$ exactly one of $a < b$, $b < a$, or $a = b$ holds (for a more detailed discussion see Barzilai [3]).

3 The Ordinal Utility Error

3.1 The Utility Property

Scale construction for physical variables requires the specification of the property which is being measured. For example, if the property under measurement is *temperature*, the construction results in a *temperature* scale and, clearly, the measurement of *length* does not produce a *mass* scale. In the case of subjective measurement too, the property under measurement must be explicitly specified and if the property under measurement is *preference*, the resulting scales are *preference* scales. Noting that von Neumann and Morgenstern's measurement of *preference* [16, §3.1] results in *utility* scales, we conclude that *preference* and *utility* (and, for the same reason, *value, worth, ophelimity,* etc.) are synonyms for the same property. It follows that the distinction between utility, preference, and value theory has no foundation in logic and science.

3.2 Mathematical Operations Are Not Applicable in Ordinal Spaces

Since order and equality are not operations, i.e. single-valued functions, no operations are defined in ordinal spaces. Specifically, the operations of addition and multiplication, and their inverses – subtraction and division, are not applicable in ordinal spaces. An ordinal space, i.e. an ordered set, is not a Euclidean space. The elementary operations of addition and multiplication are not applicable in an ordinal space and the operations and concepts of algebra and calculus are undefined in ordinal spaces. In particular, norms, metrics, derivatives, and convexity concepts are undefined and not applicable in an ordinal space. Therefore, ordinal utility functions are not differentiable and, conversely, differentiable scales cannot be ordinal and, since the partial derivatives of an ordinal utility function do not exist, the concept of marginal utility is undefined in an ordinal space.

3.3 Pareto's Claim

Pareto [17, p. 393 (translation of 1906 original)] incorrectly claims that "the entire theory of economic equilibrium is independent of the notions of (economic) *utility*" implying that ordinal utility scales are sufficient for the development of economic theory where ordinal utility is differentiated. This claim has been amplified by Hicks, Samuelson, and Debreu as follows.

3.4 Hicks's Purge

Under the titles *Need for a theory consistently based upon ordinal utility* and *The ordinal character of utility* Hicks [10, Chapter I, §§4–5] proceeds "to undertake a purge, rejecting all concepts which are tainted by quantitative utility" [10, p. 19]. In essence, he claims that wherever utility appears in economic theory, and in particular in demand theory which employs partial differentiation, it can be replaced by ordinal utility. The notion of differentiable ordinal functions is untenable and has no parallel in mathematics and science: Thermodynamics is not and cannot be founded on ordinal temperature scales. Clearly, the concept of "slope," i.e. derivative, is undefined on an *ordinal* topographic map.

3.5 Samuelson's Proof

Hicks's error, which appears in current economic textbooks, was followed in Samuelson's *Foundations of Economic Analysis* [19, pp. 94–95] by a more technical, but incorrect, argument. This analysis is carried out in an unspecified space, which in fact is an ordinal space, and operations that are not applicable in this space are applied. For example, the chain rule of differentiation is applied where the conditions for applying this rule are not satisfied. Note also that the set of ordinal scale transformations contains *all* monotone increasing functions (if $u(x)$ is an ordinal utility function, so is $F(u(x))$) where F is *any* monotone increasing function) but Samuelson's chain rule argument applies only to the subset of *differentiable* ordinal scale transformations. (Consider for example the ordinal utility function $u(x_1, x_2)$ whose value is 1 when both variables are rational and 2 otherwise.) For a detailed analysis of the ordinal utility error see Barzilai [5, §3.4].

3.6 Debreu's Existence Theorem

Adding to this argument, Debreu [8] "proved the existence of continuous utility" but, in fact, he proved the wrong theorem. What he needed to prove (and is false) is invariance with respect to *all* ordinal scale transformations. The set of ordinal scale transformations consists of all monotone increasing transformations, not just differentiable or continuous ones. Since this set contains discontinuous (and therefore non-differentiable) transformations, the set of all ordinal equivalent scales always contains discontinuous scales. This implies that continuity cannot be invariant under all ordinal (monotone increasing) scale transformations. Proving the existence of one continuous function is different from proving that all of them are continuous and, furthermore, continuity does not imply differentiability.

4 Group Decision Making

Decision theory in the presence of multiple decision makers is treated in game theory, welfare economics, and more narrowly in "Arrow's impossibility theorem."

4.1 Arrow's Non-Existence Theorem

Based on logical and technical errors, the claim that microeconomics – including welfare economics and social choice – can be founded on ordinal utility theory appears throughout the literature of modern economic theory (see e.g. Mas-Colell [13] and Mas-Colell *et al.* [14], Kreps [12], and Sen [20].) Building on this claim, Arrow [1] proceeded to show that ordinal measurement is not sufficient to construct the theory in the multiple-decision-makers case. This negative result is not surprising since no mathematical operations are applicable on ordinal scales. Of course, the notion that a constructive theory can be founded on a negative result (a non-existence theorem) is a logical error and ordinal measurement is not sufficient to construct economic theory even in the single-consumer case (see Barzilai [5, §§3.5–3.6] for details).

4.2 Game Theory

Game theory is also founded on logical and technical errors (see Barzilai [5–7] for details). The underlying utility theory is self-contradictory, addition is applied where it is inapplicable, and game theory's characteristic function is ill-defined. The concepts of von Neumann and Morgenstern's solution, imputations, and Shapley value, which depend on the characteristic function, are ill-defined as well.

In *Theory of Games and Economic Behavior* [16] von Neumann and Morgenstern proposed game theory as "the proper instrument with which to develop a theory of economic behavior." Since the operations of addition and multiplication are applicable on some scales but not on ordinal scales, some mathematical conditions must be satisfied for these operations to be applicable – addition and multiplication cannot be applied without a foundation. These conditions have not been identified and are not satisfied by any scales constructed in the classical literature, including von Neumann and Morgenstern's utility scales. For this and additional reasons, game theory cannot serve as the foundations of economic theory or other disciplines.

When the conditions for applicability of addition and multiplication on non-physical variables are satisfied, these variables are represented by points in one-dimensional affine spaces. Although vector-space operations are not applicable in affine spaces, they are applied, incorrectly, throughout the literature of economics, theory of games, decision theory, and other disciplines. For example, potential energy, which does not have an absolute zero, is an affine – rather than a vector – variable and the sum of two potential energies is undefined. The same holds for "utility" or "value" scales: the sum of "utilities" $u(x) + u(y)$ is undefined not only for different persons but also for a single person using a single fixed scale, a fact that is not recognized in the literature of welfare economics. (The operation of addition is applicable on *differences* of potential energy, time, or position.) Another example of an undefined sum appears in von Neumann and Morgenstern's definition of the characteristic function of a game [16, (25:3:c), p. 241] which is an additional error in the definition of this ill-defined function.

5   Summary

Game theory and microeconomics are founded on mathematical errors that have been committed by von Neumann and Morgenstern, Pareto, Hicks, Samuelson, and Debreu and these systematic errors have been propagated throughout the social sciences. The errors are logical and technical such as applying mathematical operations where they are not applicable, proving the wrong theorems, applying conclusions where the assumptions that lead to these conclusions are not satisfied, misidentifying the relevant mathematical spaces, using ill-defined concepts, and misinterpreting the meaning of assumptions (axioms) and conclusions.

In the case of group decision making, classical decision theory is founded on the ordinal utility error and Arrow's non-existence theorem, but decision theory is also founded on errors in the case of a single decision maker and a single criterion. The utility property is not understood (see the *utility vs. value* discussion in [5, §3.9]), neither is the measurement of preference. In fact, in the classical literature of measurement theory (e.g. Krantz *et al.* [11], Roberts [18], or Narens [15]) the mathematical models are incorrect even for the elementary properties of mass and length; the role of the property under measurement is not understood; and which operations are applicable in a given space is a question that is neither considered nor answered. For additional details see Barzilai [2–5].

References

[1]   Kenneth J. Arrow, *Social Choice and Individual Values,* Wiley, 1951.

[2]   Jonathan Barzilai, *The Challenge of Foundational Errors in Economic and Game Theory,* pp. 1 – 3. A discussion paper, Lorentz Center workshop Error in the Sciences, October 2011. Posted at www.ScientificMetrics.com

[3]   Jonathan Barzilai, *On Ordinal, Cardinal, and Expected Utility,* pp. 1–6, 2011. Posted at www.ScientificMetrics.com

[4] Jonathan Barzilai, *Correcting the Mathematical Foundations of the Social & Economic Sciences,* NSF White Paper, September 2010

[5] Jonathan Barzilai, "Preference Function Modeling: The Mathematical Foundations of Decision Theory," in *Trends in Multiple Criteria Decision Analysis,* Matthias Ehrgott, José Rui Figueira, Salvatore Greco (Eds.), Springer, pp. 57–86, 2010.

[6] Jonathan Barzilai, "Von Neumann's Error," *CORS-SCRO Bulletin*, Vol. 44 No. 4, pp. 23–24, November 2010. Posted at www.ScientificMetrics.com

[7] Jonathan Barzilai, "Value of a Game," *Notices of the American Mathematical Society,* Vol. 55 No. 4, p. 446, April 2008.

[8] Gerard Debreu, "Representation of a Preference Ordering by a Numerical Function," in *Decision processes*, R. M. Thrall, C. H. Coombs, R. L. Davis, (Eds.), Wiley, pp. 159– 165, 1954.

[9] A. Ferguson, C.S. Myers, R.J. Bartlett, H. Banister, F.C. Bartlett, W. Brown, N.R. Campbell, K.J.W. Craik, J. Drever, J. Guild, R.A. Houstoun, J.O. Irwin, G.W.C. Kaye, S.J.F. Philpott, L.F. Richardson, J.H. Shaxby, T. Smith, R.H. Thouless and W.S. Tucker, "Quantitative Estimates of Sensory Events," *Advancement of Science,* Vol. 2, pp. 331–349, 1940.

[10] John R. Hicks, *Value and Capital,* Second Edition, Oxford University Press, 1946.

[11] David H. Krantz, R. Duncan Luce, Patrick Suppes and Amos Tversky, *Foundations of Measurement,* Vol. 1, Academic Press, 1971.

[12] David M. Kreps, *A Course in Microeconomic Theory,* Princeton University Press, 1990.

[13] Andreu Mas-Colell, *The Theory of General Economic Equilibrium: A Differentiable Approach,* Cambridge University Press, 1985.

[14] Andreu Mas-Colell, Michael D. Whinston and Jerry R. Green, *Microeconomic Theory,* Oxford University Pres, 1995.

[15] Louis Narens, *Abstract measurement theory,* MIT Press, 1985.

[16] John von Neumann and Oskar Morgenstern, *Theory of Games and Economic Behavior,* Princeton University Press, 1944.

[17] Vilfredo Pareto, *Manual of Political Economy,* A.M. Kelley, 1971.

[18] Fred S. Roberts, *Measurement Theory,* Addison-Wesley, 1979.

[19] Paul A. Samuelson, *Foundations of Economic Analysis,* Harvard University Press, 1948.

[20] Amartya K. Sen, *Choice, welfare and measurement,* MIT Press, 1982.